Max-Plus Algebra Toolbox for Matlab®

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Contents

Preface

It is a short guide, to the Max–Plus Algebra Toolbox for Matlab®. This toolbox can be useful tool for calculation in the (max, +) algebra and for design and analysis of certain classes of Discrete Event Systems (DESs). The systems for which the (max, +) algebra is a convenient formalism are characterised by the aspect of synchronisation.

The Max–Plus Algebra Toolbox is a collection of functions that extend the capabilities of the Matlab® computing environment. It also correctly works under GNU Octave.

All functions in this toolbox are available in source code as M-files. The code for these functions can be viewed by using the statement:

```
>> type function_name
```

Users can extend the capabilities of this toolbox by writing their own M-files, or by using it in combination with other toolboxes. A short description of all the functions in this toolbox can be obtain by:

```
>> help function_name
```

Technical Conventions

In this contribution, we follow the (max, +) algebra notation used in [Baccelli et al. 1992].

Copyright Information

Feedback

I hope you will find this Max–Plus Algebra Toolbox helpful. If you have any suggestions or comments related to this toolbox or manual please contact. If you are using this toolbox, I would like to be informed, so, please

send me an email, I will let you know about new versions, bug fixed, updates, etc.

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The current version of this document is available on
http://gen.up.wroc.pl/stanczyk/mpa/

Installation

Installation in GNU Octave

Max-Plus Algebra Toolbox package is available as a `.tar.gz` file. Using `File Browswer` window go to the directory where the `max-plus-1.7.tar.gz` is located. Then package can be installed from the Octave prompt with the command

```
>> pkg install max-plus-1.7
```

If the package is installed successfully nothing will be printed on the prompt, but if an error occurred during installation it will be reported.
It is possible to install several package versions. If a different version of the package is already installed it will be removed prior to installing the new package. This makes it easy to upgrade and downgrade the version of a package, but makes it impossible to have several versions of the same package installed at once.

To see which packages are installed type

```
>> pkg list

-| Package Name  | Version | Installation directory
-| --------------+---------+-----------------------
-|     max-plus *|   1.7.0 | /home/js/octave/max-plus-1.7.0
```

In this case only version 1.7of the package is installed. The '*' character next to the package name shows that the image package is loaded and ready for use.

To load the package into octave workspace type

```
>> pkg load max-plus
```

To use the toolbox it is necessary to invoke GNU Octave with option `--traditional` (compatibility with Matlab®), i.e.

```
$ octave --traditional
```

1 Introduction

Many phenomena from manufacturing systems, telecommunication networks and transportation systems can be described as so-called discrete event systems (DES), or discrete event dynamic systems. A DES is a dynamic asynchronous system where the state transitions are initiated by events that occur at discrete instants of time. An event corresponds to the start or the end of an activity. A common property of such examples is that the start of an activity depends on termination of several other activities. Such systems cannot conveniently be described by differential or difference equations, and naturally exhibit a periodic behaviour.

An introduction to DES is given in e.g. [Cassandras and Lafortune 2007]. Many frameworks exist for studying DES. Examples are queueing theory, e.g. [Gross et al. 2008], Petri nets, e.g.[Murata 1989], the (max, +) algebra [Baccelli et al. 1992] and many others. The most widely used technique to analyse DES is computer simulation. Major drawback of simulation is that it often does not give a real understanding of how parameter changes affect important system properties such as stability, robustness and optimality of system performance. Analytical techniques can provide a much better insight in this respect. Therefore, formal methods are to be preferred as tools for modelling, analysis and control of DES. The theory of DES can be divided presently into three main approaches:

- *the logical approach* which considers the occurrence of events or the impossibility of this occurrence and the series of these events, but does not consider the precise time of those occurrences, e.g. an automata and formal language theory [Ramadge and Wonham 1989];
- *the quantitative approach* which addresses the issue of *performance evaluation* (evaluated by the number of events occurring in a given lapse of time), and that of *performance optimisation*, e.g. timed Petri nets or the (max, +) algebra;
- *the stochasic approach* which considers the occurrence of events under some given statistical conditions e.g. semi–Markov precesses [Limnios and Oproşan 2013].

The (max, +) algebra was first introduced in [Cuninghame–Green 1979]. A standard reference is [Baccelli et al. 1992], a brief survey of methods and applications of this algebra is given in [Cohen, Gaubert, and Quadrat 1999], [Bernd Heidergott, G. J. Olsder, and van der Woude 2005] and [De Schutter and Ton van den Boom 2008]. In certain aspects, the (max, +) algebra is comparable to the conventional algebra. In the (max, +) algebra the addition (+) and multiplication (×) operators from the conventional algebra are replaced by the maximization (max) and addition (+) operators, respectively.

Using these operators, a linear description (in the (max, +) algebra sense) of certain non–linear systems (in the conventional algebra) is achieved. Systems for which the (max, +) algebra is a proper formalism are characterized by aspect of synchronization.

In the last decade, a number of new directions appear in (max, +) systems theory, e.g. optimal control [Komenda, El Moudni, and Zerhouni 2001], [Maia et al. 2003], adaptive control [Menguy et al. 2000], stochastic control [B. Heidergott and de Vries 2001], model predictive control [T. van den Boom and De Schutter 2002], [T. J. van den Boom et al. 2003], etc.

This paper presents a software tool for rapid prototyping, modelling, control and analysis of certain classes of DESs. A Max-Plus Algebra Toolbox for Matlab [Stańczyk 2016], [Stańczyk, Mayer, and Raisch 2004] is a set of several dozen of functions implementing major aspects of the (max, +) algebra in the Matlab environment.

There are other tools available in the Internet for computation in (max, +) algebra, e.g. open-source distributed:

- *the MaxPlus Toolbox for Scilab* [MaxPlus Working Group 2003];
- *MAX*: a Maple package [Gaubert 1992];
- *C++ MinMax library*: [Gruet et al. 2015].

Main drawback of first two toolboxes presented above is that are not developed and expanded. Only C++ library is a new one and fast, but is devoted to use in C++ language.

2 (max, +) algebra

2.1 Basic operations

Definition 2.1 Max–plus algebra
The (max, +) algebra is defined as follows:

- $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the field of real numbers;
- $\forall a, b \in \mathbb{R}_\varepsilon : \quad a \oplus b \equiv \max(a, b)$;
- $\forall a, b \in \mathbb{R}_\varepsilon : \quad a \otimes b \equiv a + b$.

The algebraic structure $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called *the max-plus algebra*. We introduce the notation of $\varepsilon = -\infty$ and $e = 0$ folowing [Baccelli et al. 1992]. Notation ε and e instead of $-\infty$ and 0 respectively, is used for emphasis their special meanings and to avoid confusion with their roles in the conventional algebra. More specifically, the algebraic structure $\mathbb{R}_{\max}$ is an idempotent, commutative semiring (or dioid). This structure satisfies axioms:

- operation $\oplus$:
  - associativity:
    $$\forall a, b, c \in \mathbb{R}_\varepsilon : (a \oplus b) \oplus c = a \oplus (b \oplus c);$$
  - commutativity:
    $$\forall a, b \in \mathbb{R}_\varepsilon : a \oplus b = b \oplus a;$$
  - a neutral element (ε):
    $$\forall a \in \mathbb{R}_\varepsilon : a \oplus \varepsilon = \varepsilon \oplus a = a;$$
  - idempotent:
    $$\forall a \in \mathbb{R}_\varepsilon : a \oplus a = a;$$
- operation $\otimes$:
  - associativity:
    $$\forall a, b, c \in \mathbb{R}_\varepsilon : (a \otimes b) \otimes c = a \otimes (b \otimes c);$$

– commutative:

$$\forall a, b \in \mathbb{R}_\varepsilon : a \otimes b = b \otimes a;$$

– a neutral element (e):

$$\forall a \in \mathbb{R}_\varepsilon : e \otimes a = a \otimes e = a;$$

– an absorbing element (ε)[1]:

$$\forall a \in \mathbb{R}_\varepsilon : a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon;$$

– distributivity with respect to $\oplus$:

$$\begin{aligned} \forall a, b, c \in \mathbb{R}_\varepsilon : a \otimes (b \oplus c) &= (a \otimes b) \oplus (a \otimes c), \\ (b \oplus c) \otimes a &= (b \otimes a) \oplus (c \otimes a). \end{aligned}$$

We do not use e to denote a neutral element of operation $\otimes$, instead we use its numerical value (0) to avoid confusion with the number $e = \exp(1)$.
Note that non–zero elements do not have an inverse for $\oplus$, because $a \oplus x = b$ does not have a solution if $a > b$.
We will write ab for $a \otimes b$ whenever there is no possible confusion.

Definition 2.2 (max, +) power
Let $a \in \mathbb{R}$ and $b \in \mathbb{R}_\varepsilon$, the a–th (max, +) power of b is denoted by b^a, and corresponds to ab in conventional algebra.
If $a = 0$ then $b^a = b^0 = 0$.
If $b = \varepsilon$ and $a > 0$ then $b^a = \varepsilon^a = \varepsilon$.
If $b = \varepsilon$ and $a < 0$ then $b^a = \varepsilon^a$ is not defined.
$\varepsilon^0 = 0$ by definition.

Definition 2.3 (max, +) inversion

$$\forall a \in \mathbb{R}_\varepsilon : a^{-1} \otimes a = a \otimes a^{-1} = 0 \tag{1}$$

Definition 2.4 (max, +) division
Let $a \in \mathbb{R}_\varepsilon$ and $b \in \mathbb{R}$, the (max, +) division is defined as follows:

$$\frac{a}{b} = a \oslash b = ab^{-1}. \tag{2}$$

If $b = \varepsilon$ then $a \oslash b$ is not defined.

It is possible to derive the min operation from the (max, +) operations as follows:

$$\min(a, b) = ab \oslash (a \oplus b).$$

[1] In this convention $-\infty + \infty = -\infty$.

Table 1: Notations in the (max, +) and the conventional algebra.

(max, +) notation	conventional notation	example
$a \oplus b$	$\max(a, b)$	$3 \oplus 2 = 3$
$a \otimes b$	$a + b$	$3 \otimes 2 = 5$
a^b	$b \times a$	$3^2 = 3 \otimes 3 = 6$
$a^{1/b}$	$a \times \frac{1}{b}$	$3^{1/2} = \sqrt{3} = 1.5$
a^{-b}	$-b \times a$	$3^{-2} = -6$
$\frac{a}{b} = a \oslash b$	$a - b$	$0 \oslash 3 = -3$

Table 2: Scalar's basic functions.

operation	toolbox function				
ε	`mp_zero`	or	`mp_zeros`	or	`-Inf`
0	`mp_one`	or	`mp_ones`	or	`0`
$a \oplus b$	`mp_add(a,b)`				
$a \otimes b$	`mp_multi(a,b)`				
$a^{(-1)}$	`mp_inv(a)`	or	`mp_power(a,-1)`		
a^b	`mp_power(a,b)`				
$a^{1/b}$	`mp_power(a,1/b)`				
a^{-b}	`mp_power(a,-b)`				
$a \oslash b$	`mp_div(a,b)`	or	`mp_multi(a, mp_inv(b))`		

2.2 Matrices

Now, we extend the (max, +) algebra operations to vectors and matrices.

Definition 2.5 Scalar–vector addition
The (max, +) sum of a scalar $a \in \mathbb{R}_\varepsilon$ and a vector $\mathbf{b} \in \mathbb{R}_\varepsilon^n$ is a vector $(a \oplus \mathbf{b}) \in \mathbb{R}_\varepsilon^n$, defined by:

$$(a \oplus \mathbf{b})_i = a \oplus (\mathbf{b})_i, \qquad i = 1, \ldots, n. \tag{3}$$

A scalar–matrix addition is defined in a similar way.

Example 2.6 A (max, +) scalar-vector addition

$$4 \oplus \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \oplus 2 \\ 4 \oplus 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

Definition 2.7 Matrix addition
The sum $\oplus$ of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}_\varepsilon^{m \times n}$ is a matrix $(\mathbf{A} \oplus \mathbf{B}) \in \mathbb{R}_\varepsilon^{m \times n}$

obtained by adding corresponding entries. That is,

$$(\mathbf{A} \oplus \mathbf{B})_{ij} = (\mathbf{A})_{ij} \oplus (\mathbf{B})_{ij}, \qquad i = 1, \ldots, m;\ j = 1, \ldots, n. \tag{4}$$

Example 2.8 A (max, +) matrix addition

$$\begin{bmatrix} 1 & 6 \\ 8 & 3 \end{bmatrix} \oplus \begin{bmatrix} 2 & 5 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 \oplus 2 & 6 \oplus 5 \\ 8 \oplus 3 & 3 \oplus 3 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & 3 \end{bmatrix}.$$

Definition 2.9 Scalar–vector multiplication
We define the product of a scalar $a \in \mathbb{R}_\varepsilon$ and a vector $\mathbf{b} \in \mathbb{R}_\varepsilon^n$ as a vector $(a \otimes \mathbf{b}) \in \mathbb{R}_\varepsilon^n$:

$$(a \otimes \mathbf{b})_i = a \otimes (\mathbf{b})_i, \qquad i = 1, \ldots, n. \tag{5}$$

Scalar–matrix multiplication is defined in a similar way.

Example 2.10 A (max, +) scalar–vector multiplication

$$4 \otimes \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \otimes 2 \\ 4 \otimes 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

Definition 2.11 Matrix multiplication
The product $\otimes$ of matrices $\mathbf{A} \in \mathbb{R}_\varepsilon^{m \times p}$ and $\mathbf{B} \in \mathbb{R}_\varepsilon^{p \times n}$ is a matrix $(\mathbf{A} \otimes \mathbf{B}) \in \mathbb{R}_\varepsilon^{m \times n}$, whose (i, j)–entry is the inner product of the i^{th} row of $\mathbf{A}$ with the j^{th} column in $\mathbf{B}$. That is,

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})_{ij} &= \bigoplus_{k=1}^{p} (\mathbf{A})_{ik} \otimes (\mathbf{B})_{kj} \qquad (6) \\ &\equiv \max_k ((\mathbf{A})_{ik} + (\mathbf{B})_{kj}), \quad i = 1, \ldots, m;\ j = 1, \ldots, n, \end{aligned}$$

where: $\bigoplus_{j=1}^{m} a_j$ is short–hand for $a_1 \oplus \cdots \oplus a_m$.

A vector-matrix product is defined in a similar way.

Example 2.12 A (max, +) vector-matrix and matrix-matrix products

$$\begin{aligned} \begin{bmatrix} 2 & 8 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 \\ \varepsilon & 5 \end{bmatrix} &= \\ \begin{bmatrix} (2 \otimes 2) \oplus (8 \otimes \varepsilon) & (2 \otimes 0) \oplus (8 \otimes 5) \end{bmatrix} &= \\ \begin{bmatrix} 4 \oplus \varepsilon & 2 \oplus 13 \end{bmatrix} &= \begin{bmatrix} 4 & 13 \end{bmatrix}; \end{aligned}$$

c)

$$\begin{bmatrix} 1 & 6 & 2 \\ 8 & 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 2 & 5 \\ 3 & 3 \\ 1 & 6 \end{bmatrix} =$$

$$\begin{bmatrix} (1 \otimes 2) \oplus (6 \otimes 3) \oplus (2 \otimes 1) & (1 \otimes 5) \oplus (6 \otimes 3) \oplus (2 \otimes 6) \\ (8 \otimes 2) \oplus (3 \otimes 3) \oplus (4 \otimes 1) & (8 \otimes 5) \oplus (3 \otimes 3) \oplus (4 \otimes 6) \end{bmatrix} =$$

$$\begin{bmatrix} 3 \oplus 9 \oplus 3 & 6 \oplus 9 \oplus 8 \\ 10 \oplus 6 \oplus 5 & 13 \oplus 6 \oplus 10 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 10 & 13 \end{bmatrix}.$$

Definition 2.13 Identity matrix
The matrix $\mathbf{I}_n \in \mathbb{R}_\varepsilon^{n \times n}$ with 0's on the main diagonal and ε's elsewhere is called the identity matrix of order n.

Identity matrix is a neutral element for matrices $\otimes$ operation:

$$\mathbf{A} \otimes \mathbf{I} = \mathbf{A} \tag{7}$$

Definition 2.14 Zero matrix
The matrix $\boldsymbol{\varepsilon}^{m \times n}$ with $\left(\boldsymbol{\varepsilon}\right)_{ij} = \varepsilon$ for all i, j, is the zero–matrix.

Zero matrix is a neutral element for matrices $\oplus$ operation:

$$\mathbf{A} \oplus \boldsymbol{\varepsilon} = \mathbf{A} \tag{8}$$

and

$$\mathbf{A} \otimes \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \tag{9}$$

Definition 2.15 (max, +) matrix power
Let $\mathbf{A} \in \mathbb{R}_\varepsilon^{n \times n}$ and $b \in \mathbb{N}_0$, the b–th (max, +) power of $\mathbf{A}$ is defined as follows:

$$\mathbf{A}^b = \underbrace{\mathbf{A} \otimes \cdots \otimes \mathbf{A}}_{b} = \bigotimes_{i=1}^{b} \mathbf{A}. \tag{10}$$

If $b = 0$ then $\mathbf{A}^0 = \mathbf{I}$.
If $b = -1$ see theorem 2.16.
If $b \notin \mathbb{N}_0 \cup \{-1\}$ then $\mathbf{A}^b$ is not defined.
$\mathbb{N}_0$ is the set of nonnegative integers.

Theorem 2.16 *[Cuninghame–Green 1979]*
A matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ is invertible in the (max, +) algebra if and only if it can be factorized as

$$\mathbf{A} = \mathbf{DP}, \tag{11}$$

where:

$\mathbf{D} \in \mathbb{R}_\varepsilon^{n\times n}$ is a matrix with non-εdiagonal entries
$\mathbf{P} \in \mathbb{R}_\varepsilon^{n\times n}$ is a permutation matrix.

Then

$$\mathbf{A}^{-1} = \mathbf{P}^{-1}\mathbf{D}^{-1}$$

where:

$$(\mathbf{D}^{-1})_{ii} = -(\mathbf{D})_{ii},$$
$$\mathbf{P}^{-1} = \mathbf{P}^T.$$

Example 2.17 *Matrix inversion*
Consider $\mathbf{A} = \begin{bmatrix} \varepsilon & \varepsilon & 3 \\ -2 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$.
This matrix is (max, +)–invertible:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \varepsilon & \varepsilon & 3 \\ -2 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix} \\ &= \mathbf{DP} \\ &= \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{bmatrix} \begin{bmatrix} \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{bmatrix} \end{aligned}$$

We have

$$\begin{aligned} \mathbf{A}^{-1} &= \mathbf{P}^{-1}\mathbf{D}^{-1} \\ &= \begin{bmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ -3 & \varepsilon & \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ -3 & \varepsilon & \varepsilon \end{bmatrix}. \end{aligned}$$

Lets check:

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{A} &= \begin{bmatrix} \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ -3 & \varepsilon & \varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon & \varepsilon & 3 \\ -2 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{bmatrix} \\ &= \mathbf{I}\end{aligned}$$

Example 2.18 Not inversible matrix
Consider $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$.

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{A} &= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \\ &\neq \mathbf{I}\end{aligned}$$

Definition 2.19 (max, +) matrix division
Let $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}_{\varepsilon}^{n\times n}$.
If $\mathbf{B}$ is (max, +)–invertible then the (max, +) matrix division $\mathbf{A}$ by $\mathbf{B}$ is defined as follows:

$$\mathbf{A} \oslash \mathbf{B} = \mathbf{B}^{-1}\mathbf{A}. \tag{12}$$

Definition 2.20 Trace of a matrix
The *trace* of a matrix $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$ is the sum of the entries on its main diagonal and is denoted by $tr(\mathbf{A})$. That is,

$$tr(\mathbf{A}) = \bigoplus_{i=1}^{n} (\mathbf{A})_{ii}. \tag{13}$$

Equivalently, $tr(\mathbf{A})$ is a maximal element from main diagonal.

Table 3: Functions to operate on matrices.

function	short description	
`mp_zeros(n,m)`	n–by–m zeros matrix	
`mp_ones(n,m)`	n–by–m ones matrix	
`mp_eye(n,m)`	n–by–m identity matrix	$\mathbf{I}_n$
`mp_add(A,B)`	addition of $\mathbf{A}$ and $\mathbf{B}$	$\mathbf{A} \oplus \mathbf{B}$
`mp_multi(A,B)`	multiplication of $\mathbf{A}$ by $\mathbf{B}$	$\mathbf{A} \otimes \mathbf{B}$
`mp_power(A,n)`	n–th power of $\mathbf{A}$	$\mathbf{A}^n$
`mp_inv(A)`	invertion of $\mathbf{A}$	$\mathbf{A}^{-1}$
`mp_div(A,B)`	division of $\mathbf{A}$ by $\mathbf{B}$	$\mathbf{B}^{-1}\mathbf{A}$
`mp_trace(A)`	trace of $\mathbf{A}$	$tr(\mathbf{A})$

2.3 Connection with graph theory

There exists a close relation between the (max, +) algebra and graphs. In this subsection we first give a short introduction to graph theory, and next we give a graph–theoretic interpretation of some basic (max, +) operations and concepts, which will be used later on. For the introduction to the graphs we refer the reader to e.g. [Aldous and Wilson 2000].

Definition 2.21 graph
A *graph* $\mathcal{G}$ is defined as a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of elements called *nodes* (or *vertices*) and $\mathcal{E}$ is a set the elements called *edges*. Each edge joints two nodes.

Definition 2.22 digraph
A *directed graph* (or shorter *digraph*) $\mathcal{G}$ is defined as a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of elements called *nodes* (or *vertices*) and $\mathcal{E}$ is a set the elements which are ordered pairs of nodes called *arcs*.

Definition 2.23 weighted digraph
A digraph is called *weighted digraph* if there exists associated element a_{ij} with each arcs $(j, i) \in \mathcal{E}$. The quantity a_{ij} is called the *weight* of arc (j, i).

Definition 2.24 initial and final node, predecessor and successor
Denote the number of nodes by n and number of the individual nodes $1, 2, \ldots, n$.
If a pair $(i, j) \in \mathcal{E}$, then i is called *initial node* (or *origin*) of the arc (i, j), and j the *final node* (or *destination*) of arc (i, j).
If a pair $(i, j) \in \mathcal{E}$, then i is called *predecessor* of j and j is called

successor of i. The set of all predecessors of j is indicated by $\pi(j)$.

Definition 2.25 path, circuit and acyclic digraph
A *path* ρ is a sequence of nodes $(i_1, i_2, \ldots, i_p)$, $p > 1$, such that $i_j \in \pi(i_{j+1})$, $j = 1, \ldots, p-1$.
Equivalently, a path is a sequence of arcs which connects a sequence of nodes.
A *circuit* is a path, in which the initial and final nodes coincide.
A digraph is *acyclic* if it does not contain circuits.

Definition 2.26 lenght and weight of a path
A *length* of a path (or circuit) $|\rho|_l$ is equal to the sum of the lengths of the arcs of which it is composed.
A *weight* of a path (or circuits) $|\rho|_w$ is the sum of the weights of the individual arcs.

Definition 2.27 connected and strongly connected digraph
A digraph is *connected* if its underlying graph is a connected graph (i.e. there exists a path between each pair of vertices), and is *disconnected* otherwise.
A digraph is *strongly connected* if there is a path between each pair of vertices.

Fig. 1 shows a connected but not strongly connected digraph.

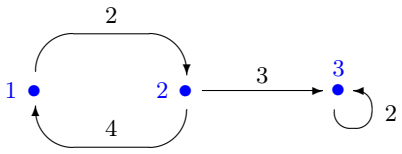

Figure 1: A (not strongly) connected digraph.

Definition 2.28 Precedence graph
The *precedence graph* $\mathcal{G}(\mathbf{A})$ of matrix $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$ is a weighted digraph with n nodes and arc (j, i) if and only if $(\mathbf{A})_{ij} \neq \varepsilon$.

Theorem 2.29
Let $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$ and let $\mathbf{B}$ be a matrix

$$\mathbf{B} = \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^n, \tag{14}$$

then $\mathcal{G}(\mathbf{A})$ is strongly connected if and only if each entry in $\mathbf{B}$ is greater than ε.

The proof foolows straight from the standard version, we refer the reader to e.g. [Aldous and Wilson 2000].

Definition 2.30 Maximum cycle mean
The *maximum cycle mean* of precedence graph $\mathcal{G}(\mathbf{A})$ is the maximum of the average weight of circuits

$$\lambda = \max_{\rho} \frac{|\rho|_w}{|\rho|_l}, \tag{15}$$

where:
$|\rho|_l$ is the length and $|\rho|_w$ is the weight of the circuit.

It can be rewritten in (max, +) notation:

$$\lambda = \bigoplus_{j=1}^{n} tr(\mathbf{A}^j)^{\frac{1}{j}}, \tag{16}$$

where:
$(\mathbf{A}^j)_{ii}$ — the maximum weight of all circuits of length j which pass through node i,
$tr(\mathbf{A}^j)$ — maximum over all nodes.

a)

$$\mathbf{A} = \begin{bmatrix} 5 & \varepsilon & 5 \\ \varepsilon & 6 & 3 \\ 11 & 12 & 11 \end{bmatrix}$$

b)

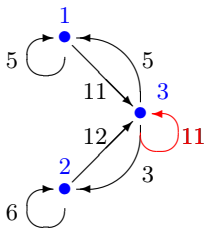

Figure 2: An exemplary matrix (a) and its precedence graph (b) with maximum cycle mean equals 11.

Theorem 2.31
[Karp 1978] a bit modified by [Gaubert and Scilab 1998].
If precedence graph of $\mathbf{A}$ is strongly connected (or if $\mathbf{A}$ irreducible), then

$$\lambda = \max_{\substack{1\leqslant j\leqslant n \\ (\mathbf{A}^n)_{ij}\neq -\infty}} \min_{1\leqslant k\leqslant n} \frac{(\mathbf{A}^n)_{ij} - (\mathbf{A}^{n-k})_{ij}}{k}. \tag{17}$$

So, in the (max, +) notation

$$\lambda = \bigoplus_{\substack{1\leqslant j\leqslant n \\ (\mathbf{A}^n)_{ij}\neq \varepsilon}} \bigvee_{1\leqslant k\leqslant n} \left(\frac{(\mathbf{A}^n)_{ij}}{(\mathbf{A}^{n-k})_{ij}} \right)^{\frac{1}{k}}. \tag{18}$$

where:

$a \vee b \equiv \min(a, b)$.

A good bibliography on the maximal cycle mean problem, and a comparison of Karp's algorithm with other classical algorithms, can be found in [Dasdan and Gupta 1998].

Table 4: Graph functions.

function	short description
`mp_is_pga(A)`	checks, if a precedence graph $\mathcal{G}(\mathbf{A})$ is acyclic
`mp_is_pgc(A)`	checks, if a precedence graph $\mathcal{G}(\mathbf{A})$ is connected
`mp_is_pgsc1(A)`	checks, if a precedence graph $\mathcal{G}(\mathbf{A})$ is strongly connected (from definition 2.27)
`mp_is_pgsc2(A)`	checks, if a precedence graph $\mathcal{G}(\mathbf{A})$ is strongly connected (from theorem 2.29)
`mp_mcm(A)`	maximum cycle mean of $\mathcal{G}(\mathbf{A})$ (from (16))
`mp_mcm_fw(A)`	maximum cycle mean of $\mathcal{G}(\mathbf{A})$ Floyd–Warshall algorithm [Floyd 1962]
`mp_mcp_karp(A)`	maximum cycle mean of $\mathcal{G}(\mathbf{A})$ Karp algorithm: 2.31

2.4 Linear equations

Some of the examples in this chapter are taken from [Gaubert and Scilab 1998].

2.4.1 Problem Ax = b

(max, +) algebra is equipped with the following natural order relation:

$$\forall a, b \in \mathbb{R}_\varepsilon : a \preccurlyeq b \iff a \oplus b = b. \tag{19}$$

Let us consider equation (20). Let $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}_{\varepsilon}^{n}$:

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{20}$$

Generally (20) has no solution, but (21) always does.

$$\mathbf{A}\mathbf{x} \preccurlyeq \mathbf{b}. \tag{21}$$

We can try solve it via residuation [Blyth and Janowitz 1972]. Precisely, this method solves (21).

Theorem 2.32 [Baccelli et al. 1992]
Given $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}_{\varepsilon}^{n}$, the greatest subsolution of (20) exists, and is given by

$$(\mathbf{i} \oslash \mathbf{x})^T = (\mathbf{i} \oslash \mathbf{b})^T \mathbf{A}, \tag{22}$$

where $\mathbf{i}$ is (max, +)–algebraic identity vector with appropriate size.

Let solution of (22) denote as

$$\mathbf{x} = (\mathbf{A} \backslash \mathbf{b}). \tag{23}$$

Equation (20) has a solution if and only if

$$\mathbf{A}(\mathbf{A} \backslash \mathbf{b}) = \mathbf{b}. \tag{24}$$

Table 5: Toolbox functions assigned to the problem $\mathbf{A}\mathbf{x} = \mathbf{b}$.

function	short description
`mp_solve_Axb(A,b)`	the greatest subsolution of $\mathbf{A}\mathbf{x} = \mathbf{b}$

2.4.2 Problem $\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{b}$

Theorem 2.33 [Baccelli et al. 1992]
Let $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$, and $\mathbf{b} \in \mathbb{R}_{\varepsilon}^{n}$. The minimal column vector $\mathbf{x} \in \mathbb{R}_{\varepsilon}^{n}$, such that

$$\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{b} \tag{25}$$

is given by

$$\mathbf{x} = \mathbf{A}^{\star}\mathbf{b}. \tag{26}$$

Definition 2.34 Star operator
The operator * for square matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ is defined by:

$$\mathbf{A}^\star = \bigoplus_{k\in\mathbb{N}_0} \mathbf{A}^k \tag{27}$$

where:

$$\mathbf{A}^0 = \mathbf{I}_n,\ \mathbf{A}^k = \mathbf{A} \otimes \mathbf{A}^{k-1},$$

$\mathbb{N}_0$ is the set of nonnegative integers.

We can interpret $(\mathbf{A}^\star)_{ij}$ as the maximal weight of a path from i to j of any lenght, in the $\mathcal{G}(\mathbf{A})$. $\mathbf{A}^\star$ is a priori defined in $(\mathbb{R}_\varepsilon \cup \{+\infty\})^{n\times n}$, but the $+\infty$ value is undesired in most application.

Theorem 2.35 [Gaubert 1997]
Let $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$, iff there are no circuits with positive weight in precedence graph $\mathcal{G}(\mathbf{A})$, then

$$\mathbf{A}^\star = \mathbf{A}^0 \oplus \mathbf{A}^1 \oplus \cdots \oplus \mathbf{A}^{n-1}. \tag{28}$$

Table 6: Toolbox functions assigned to the problem $\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{b}$.

function	short description	
`mp_star(A)`	(max, +) star operator	$\mathbf{A}^\star$
`mp_solve_xAxb(A,b)`	the solution of $\mathbf{x} = \mathbf{A}\mathbf{x} \oplus \mathbf{b}$	$\mathbf{x} = \mathbf{A}^\star\mathbf{b}$

2.4.3 Spectral problem $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

The most useful (max, +) practicable results are related to the spectral problem (29).

Definition 2.36 The (max, +) spectral problem
The matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ has an eigenvalue in the (max, +) algebra sense, if there exist a real number $\lambda \in \mathbb{R}$ and a vector $\mathbf{v} \in \mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}. \tag{29}$$

The vector $\mathbf{v}$ is then called an eigenvector for the eigenvalue λ.

The theory is extremely similar to the well known Perron–Frobenius theory (see e.g. [Bapat 1998]). Every square matrix with entries in $\mathbb{R}_\varepsilon$ has at least one eigenvalue. Unlike in conventional Perron–Frobenius theory, an

irreducible matrix can have several (non proportional) eigenvectors.

(max, +) algebraic eigenproblem has been studied in many publications (see e.g. [Baccelli et al. 1992] and in particular [Braker 1993] for a power algorithm). In some publications, authors allows components of vector $\mathbf{v}$ in the eigenvalue problem to assume the value ε, with the exception of (max, +) zero vector — a trivial solution of (29).

Determine an eigenvalue

Theorem 2.37 [Baccelli et al. 1992]
If $\mathbf{A}$ is irreducible, or equivalently if $\mathcal{G}(\mathbf{A})$ is strongly connected, there exists one, and only one eigenvalue (but possibly several eigenvectors). This eigenvalue is equal to the maximum cycle mean of $\mathcal{G}(\mathbf{A})$.

If $\mathcal{G}(\mathbf{A})$ is not strongly connected then we can find the maximum cycle mean by determining the maximum cycle mean for each strong component of $\mathcal{G}(\mathbf{A})$. The strong components can be found using algorithm presented in [Tarjan 1972].

Lemma 2.38 [G.-J. Olsder, Roos, and van Egmond 1999]
A solution of eigenproblem (29) is feasible if and only if $\mathcal{G}(\mathbf{A})$ contains a circuit.

Example 2.39 Eigenvalue of not irreducible matrix

- The following example shows the uniqueness of the eigenvalue when $\mathcal{G}(\mathbf{A})$ is not connected:

$$\begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} = 1 \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix},$$

$$\begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 2 \end{bmatrix} \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} = 2 \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 2 \end{bmatrix}.$$

- In the following example $\mathcal{G}(\mathbf{A})$ is connected but not strongly connected, and there are two eigenvalues:

$$\begin{bmatrix} 0 & 0 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} = 0 \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We can also use an another way to obtain the eigenvalue. The starting point is the equation of linear system (30), which describes the time evolution of an autonomous DES (for more details see § 4):

$$\mathbf{x}(k) = \mathbf{A}\mathbf{x}(k-1), \tag{30}$$

where $\mathbf{x} \in \mathbb{R}_{\varepsilon}^{n}$ and $\mathbf{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$.

Theorem 2.40
Given an initial vector $\mathbf{x}(0)$ (the initial state of the system), we say that the system (30), after a number of steps, ends up in a periodic behaviour, if there exists integers p, q with $p > q \geqslant 0$ and real number c such that $\mathbf{x}(p) = \mathbf{x}(q) \oplus c$. Then the eigenvalue λ is given by

$$\lambda = \frac{c}{p-q} \qquad \text{(in the conventional algebra).} \tag{31}$$

Example 2.41
Let $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$ then $\mathbf{x}(1) = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \rightarrow \mathbf{x}(2) = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \rightarrow \mathbf{x}(3) = \begin{bmatrix} 16 \\ 13 \end{bmatrix}$. So, $\mathbf{x}(3) - \mathbf{x}(1) = \begin{bmatrix} 9 \\ 9 \end{bmatrix}$. Hence $\lambda = \dfrac{9}{3-1} = 4.5$.

Looking for eigenvectors: implemented algorithms

Algorithm 1 [G.-J. Olsder 1991]
Candidate eigenvector — first so-called power algorithm.
If we have p, q and c (see theorem 2.40), then the eigenvector is given by:

$$\mathbf{v} = \frac{1}{p-q} \sum_{j=1}^{p-q} \mathbf{x}(q+j-1) \qquad \text{(in the conventional algebra).} \tag{32}$$

Note: This algorithm does not always give an eigenvector.

Example 2.42
Let $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$. So $p = 3, q = 1, c = 9$.
Hence $\mathbf{v} = \dfrac{1}{3-1}\big(\mathbf{x}(1) + \mathbf{x}(2)\big) = \dfrac{1}{2}\Big(\begin{bmatrix} 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 11 \\ 9 \end{bmatrix}\Big) = \begin{bmatrix} 9 \\ 6.5 \end{bmatrix}$.

Algorithm 2 [Braker and G.-J. Olsder 1993]
Extended alg. 1 — power algorithm.

1. Calculate a candidate for an eigenvector from (32);
2. If $\mathbf{v}$ is an eigenvector then stop, if not, go to step 3;
3. Define a new vector:

$$\overline{v_i} = \begin{cases} v_i, & \text{if } (\mathbf{A} \otimes \mathbf{v})_i = \lambda \otimes v_i \\ \varepsilon, & \text{elsewhere} \end{cases} \tag{33}$$

4. Restart algorithm with $\mathbf{x}(0) = \overline{\mathbf{v}}$,
   until for some $r \geqslant 0$, there holds $\mathbf{x}(r+1) = \lambda \otimes \mathbf{x}(r)$,
   then $\mathbf{x}(r)$ is an eigenvector of $\mathbf{A}$.

Example 2.43

Let $\mathbf{A} = \begin{bmatrix} \varepsilon & 3 & \varepsilon & 1 \\ 2 & \varepsilon & 1 & \varepsilon \\ 1 & 2 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{bmatrix}$, and $\mathbf{x}(0) = \begin{bmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$, so $\lambda = 2\frac{1}{2}$.

A result form alg. 2 is $\mathbf{v} = \begin{bmatrix} 5 \\ 4\frac{1}{2} \\ 5 \\ 3\frac{1}{2} \end{bmatrix}$.

It is easy to see that $\mathbf{A} \otimes \mathbf{v} = \begin{bmatrix} 7\frac{1}{2} \\ 7 \\ 7 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 7\frac{1}{2} \\ 7 \\ 7\frac{1}{2} \\ 6 \end{bmatrix} = \lambda \otimes \mathbf{v}$.

The new vector $\overline{\mathbf{v}} = \begin{bmatrix} 5 \\ 4\frac{1}{2} \\ \varepsilon \\ 3\frac{1}{2} \end{bmatrix} = \mathbf{x}(0)$, and then

$$\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4\frac{1}{2} \\ \varepsilon \\ 3\frac{1}{2} \end{bmatrix} \to \mathbf{x}(1) = \begin{bmatrix} 7\frac{1}{2} \\ 7 \\ 6\frac{1}{2} \\ \varepsilon \end{bmatrix} \to \mathbf{x}(2) = \begin{bmatrix} 10 \\ 9\frac{1}{2} \\ 9 \\ 7\frac{1}{2} \end{bmatrix} \to \mathbf{x}(3) = \begin{bmatrix} 12\frac{1}{2} \\ 12 \\ 11\frac{1}{2} \\ 10 \end{bmatrix} = \lambda \otimes \mathbf{x}(2).$$

Hence, a result of alg. 2 is $\begin{bmatrix} 10 \\ 9\frac{1}{2} \\ 9 \\ 7\frac{1}{2} \end{bmatrix}$.

Algorithm 3 [G.-J. Olsder, Roos, and van Egmond 1999]
With adapted Floyd–Warshall procedure [Ahuja, Magnanti, and Orlin 1993].
Let $\mathcal{G}(\mathbf{A}) = (\mathcal{V}, \mathcal{E})$ be a precedence graph of $\mathbf{A}$, $\mathbf{J}$ denotes the all–one matrix.

1. We start by choosing a lower bound μ for λ.
   For this we can take $\mu = \max_{i\in\mathcal{V}}(\mathbf{A})_{ii}$ if the diagonal of $\mathbf{A}$ is not void; otherwise we may use $\mu = \min_{i,j\in\mathcal{E}}(\mathbf{A})_{ij}$, because the average weight of a circuit cannot be smaller.

2. Then we check (see below) whether there exists a circuit C with positive weight with respect to $\mathbf{A} - \mu\mathbf{J}$. If this is the case we increase μ to the average weight of C and restart the procedure. This is repeated until there are no circuits having positive weight with respect to $\mathbf{A}' = \mathbf{A} - \mu\mathbf{J}$. It implies that $\lambda = \mu$.

   The (adapted) Floyd–Warshall procedure to detect the existence of circuits having positive weight goes as follows. Starting with $\mathbf{A}_0 = \mathbf{A}$ we successively construct matrices $\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n$ by defining

   $$(\mathbf{A}_k)_{ij} = \max\big((\mathbf{A}_{k-1})_{ij}, (\mathbf{A}_{k-1})_{ik} + (\mathbf{A}_{k-1})_{kj}\big), \qquad k = 1, 2, ..., n. \tag{34}$$

   If we apply this procedure to the matrix $\mathbf{A}' = \mathbf{A} - \mu\mathbf{J}$ and

   - if there exists no positive circuits with respect to $\mathbf{A}' = \mathbf{A} - \mu\mathbf{J}$ (i.e. when $\mu = \lambda$), then the nodes lying on a critical circuit are exactly those i for which $(\mathbf{A}')_{ii} = 0$;
   - if there exists a positive circuits with respect to $\mathbf{A}'$ (i.e., when $\mu < \lambda$), the existence of such a circuit is detected by the Floyd–Warshall procedure as soon as a diagonal entry in one of the matrices $\mathbf{A}'_k$ becomes positive. This circuit, C say, can be found by backtracking the Floyd–Warshall procedure, starting from the node corresponding to the positive diagonal entry. We replace μ by the average weight of C with respect to $\mathbf{A}$ and restart the Floyd–Warshall procedure on the updated matrix $\mathbf{A} - \mu\mathbf{J}$.

3. The eigenvectors can be immediately obtained from the final matrix $\mathbf{A}'_n$ generated by the Floyd–Warshall procedure. This matrix reveals all critical circuits and all nodes lying on critical circuits; these are precisely the nodes whose columns in $\mathbf{A}'_n$ have a zero on the diagonal. We denote the critical circuits by C_k where k runs through an index set K whose cardinality is equal to the number of critical circuits. For each $k \in K$ we choose a node r_k on C_k. We call r_k the reference node for the critical circuit C_k. Moreover, for each node i we define

   $$x_i^{(k)} = \text{maximum weight of all paths in } \mathcal{G}(\mathbf{A}') \text{ from } r_k \text{ to } i,\ i \in \mathcal{V}.$$

   If there exists no path from r_k to i then the corresponding entry of $x^{(k)}$ will be taken ε. Note that $x^{(k)}$ is just the column in the matrix $\mathbf{A}'_n$ corresponding to the node r_k.

Now let L be any nonempty subset of K. For any such subset we define

$$x_i^L = \max\{x_i^{(k)} : \ k \in L\}, i \in \mathcal{V}. \tag{35}$$

If L is such that the vector $\mathbf{x}^L$ is finite then $\mathbf{x}^L$ is a solution of the eigenvalue problem (29).

Example 2.44
Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 \\ 2 & 5 & 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 8 & 3 & \varepsilon & \varepsilon \\ 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 8 & 3 \end{bmatrix}.$$

As a first guess for the maximum cycle mean of $\mathcal{G}(\mathbf{A})$ we take the maximum entry on the diagonal of $\mathbf{A}$, thus $\mu = (\mathbf{A})_{22} = 5$. Next we apply the Floyd–Warshall procedure to the matrix $\mathbf{A} - 5\mathbf{J}$. After three iterations we find a positive entry on the diagonal:

$$(\mathbf{A} - 5\mathbf{J})_3 = \begin{bmatrix} -1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 \\ -3 & 0 & -4 & -3 & \varepsilon & -4 \\ \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & 4 & \varepsilon & \varepsilon \\ 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & -2 \end{bmatrix}.$$

There is a circuit containing node 4 of weight 4. Indeed we have a circuit $4 \to 3 \to 4$ which has circuit mean 7 with respect to $\mathbf{A}$. We therefore proceed with $\mu = 7$ and repeat the Floyd–Warshall procedure to the matrix $\mathbf{A} - 7\mathbf{J}$. This time all diagonal entries remain nonpositive:

$$(\mathbf{A} - 7\mathbf{J})_6 = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & -2 & -3 \\ -5 & -2 & -6 & -7 & -7 & -8 \\ \varepsilon & \varepsilon & 0 & -1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & 0 & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon & 0 & -1 \\ 3 & \varepsilon & \varepsilon & \varepsilon & 1 & 0 \end{bmatrix}.$$

The zero diagonal entries of $(\mathbf{A} - 7\mathbf{J})_6$ give us the critical nodes.
It turns out that $\mathcal{G}(\mathbf{A})$ has two critical circuits: $4 \to 3 \to 4$ and $1 \to 5 \to 6 \to$

1 with $\mu = 7$. Application of algorithm yields the following six eigenvectors ($\mathbf{x}^{\{6,4\}}$ is a linear combination of $\mathbf{x}^{\{5,3\}}$):

$$\mathbf{x}^{\{1,3\}} = \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}^{\{5,3\}} = \begin{bmatrix} -2 \\ -6 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}^{\{6,3\}} = \begin{bmatrix} -3 \\ -6 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

$$\mathbf{x}^{\{1,4\}} = \begin{bmatrix} 0 \\ -5 \\ -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}^{\{5,4\}} = \begin{bmatrix} -2 \\ -7 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}^{\{6,4\}} = \begin{bmatrix} -3 \\ -7 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Algorithm 4 [Subiono and van der Woude 2000]
Extended alg. 1 and 2.

1. Calculate a candidate for an eigenvector from (32);
2. If $\mathbf{v}$ is an eigenvector then stop, if not, go to step 3;
3. Restart algorithm with $\mathbf{x}(0) = \mathbf{v}$,
   until for some $r \geqslant 0$, there holds $\mathbf{x}(r+1) = \lambda \otimes \mathbf{x}(r)$,
   then $\mathbf{x}(r)$ is an eigenvector of $\mathbf{A}$.

Example 2.45

Let $\mathbf{A} = \begin{bmatrix} \varepsilon & 3 & \varepsilon & 1 \\ 2 & \varepsilon & 1 & \varepsilon \\ 1 & 2 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{bmatrix}$, and an initial state $\mathbf{x}(0) = \begin{bmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$, so $\lambda = 2\frac{1}{2}$.

A result form alg. 1 is $\mathbf{v} = \begin{bmatrix} 5 \\ 4\frac{1}{2} \\ 5 \\ 3\frac{1}{2} \end{bmatrix}$.

We can see that $\mathbf{A} \otimes \mathbf{v} = \begin{bmatrix} 7\frac{1}{2} \\ 7 \\ 7 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 7\frac{1}{2} \\ 7 \\ 7\frac{1}{2} \\ 6 \end{bmatrix} = \lambda \otimes \mathbf{v}$.

So, the new vector $\mathbf{x}(0) = \mathbf{v}$, and:

$$\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4\frac{1}{2} \\ 5 \\ 3\frac{1}{2} \end{bmatrix} \rightarrow \mathbf{x}(1) = \begin{bmatrix} 7\frac{1}{2} \\ 7 \\ 7 \\ 6 \end{bmatrix} \rightarrow \mathbf{x}(2) = \begin{bmatrix} 10 \\ 9\frac{1}{2} \\ 9 \\ 8 \end{bmatrix} \rightarrow$$

$$\rightarrow \mathbf{x}(3) = \begin{bmatrix} 12\frac{1}{2} \\ 12 \\ 11\frac{1}{2} \\ 10 \end{bmatrix} \rightarrow \mathbf{x}(4) = \begin{bmatrix} 15 \\ 14\frac{1}{2} \\ 14 \\ 12\frac{1}{2} \end{bmatrix} = \lambda \otimes \mathbf{x}(3).$$

Hence, a result of alg. 4 is $\begin{bmatrix} 12\frac{1}{2} \\ 12 \\ 11\frac{1}{2} \\ 10 \end{bmatrix}$.

Algorithm 5 [Subiono and van der Woude 2000]
If we have p, q and c (see theorem 2.40), then the eigenvector is given by:

$$\mathbf{v} = \bigoplus_{j=1}^{p-q} (\lambda^{p-q-j} \otimes \mathbf{x}(q+j-1)). \tag{36}$$

Example 2.46
Let $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$. So $p = 3, q = 1, c = 9$ and $\lambda = 4.5$.
Hence $\mathbf{v} = \lambda^1 \mathbf{x}(1) \oplus \lambda^0 \mathbf{x}(2) = 4.5 \begin{bmatrix} 7 \\ 4 \end{bmatrix} \oplus 1 \begin{bmatrix} 11 \\ 9 \end{bmatrix} = \begin{bmatrix} 11.5 \\ 9 \end{bmatrix}$.

Table 7: Toolbox functions assign to the spectral problem $\mathbf{Av} = \lambda\mathbf{v}$.

function	short description
mp_is_pga	checks if A has got any eigenvalue
mp_is_pgsc1	checks if A has got exactly one eigenvalue
mp_is_pgsc2	checks if A has got exactly one eigenvalue
mp_is_egv1	checks if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$
mp_is_egv2	checks if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$
mp_pqc	calculate components of an eigenvalue of $\mathbf{A}$, from (31)
mp_egv_pqc	eigenvalue of $\mathbf{A}$, from (31)
mp_mcm	eigenvalue of $\mathbf{A}$, from (16)
mp_mcm_fw	eigenvalue of $\mathbf{A}$ - Floyd–Warshall algorithm
mp_mx_fw	final Floyd–Warshall matrix of $\mathbf{A}$
mp_ev_fw	eigenvectors of $\mathbf{A}$ - Floyd–Warshall algorithm
mp_mcm_karp	eigenvalue of $\mathbf{A}$ - Karp algorithm
mp_egv_o91	eigenvalue and eigenvector of $\mathbf{A}$, [G.-J. Olsder 1991]
mp_egv_bo93	eigenvalue and eigenvector of $\mathbf{A}$, [Braker and G.-J. Olsder 1993]
mp_egv_sw001	eigenvalue and eigenvector of $\mathbf{A}$, [Subiono and van der Woude 2000]
mp_egv_sw002	eigenvalue and eigenvector of $\mathbf{A}$, [Subiono and van der Woude 2000]

3 A bit of the (min, +) algebra

Definition 3.1 The (min, +) algebra
(min, +) algebra is defined as follows:

- $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{+\infty\}$, where $\mathbb{R}$ is the field of real numbers;
- $\forall a, b \in \mathbb{R}_\varepsilon : \quad a \oplus b \equiv \min(a, b)$;
- $\forall a, b \in \mathbb{R}_\varepsilon : \quad a \otimes b \equiv a + b$.

The algebraic structure $\mathbb{R}_{\min} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, is called *the min-plus algebra*. The reason for not distinguishing between the maximum in (max, +) and the minimum in (min, +) operations is that $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are isomorphic algebraic structures and such notations are frequently used in literature. But in this paper to avoid any ambiguity we will use different symbols:

- $a \vee b \equiv \min(a, b)$;
- $a \wedge b \equiv a + b$.

We decide to use different symbol for + in (max, +) and in (min, +), because

$$-\infty \otimes \infty = -\infty \neq -\infty \wedge \infty = \infty. \tag{37}$$

The (max, +) and (min, +) algebra together comprise the *minimax algebra* [Cuninghame–Green 1979].

The basic scalar, vector and matrix operations in the (min, +) are formed analogously with the (max, +), and they are ommited here.

Definition 3.2 Plus operator
The operator $^+$ for square matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{n \times n}$ is defined by:

$$\mathbf{A}^+ = \bigvee_{k \in \mathbb{N}} \mathbf{A}^k, \tag{38}$$

where $\mathbb{N}$ is the set of natural numbers.

It does mean, that

$$\mathbf{A}^+ \vee \mathbf{A}^0 = \mathbf{A}^\star. \tag{39}$$

The matrix $\mathbf{A}^+$ sometimes referred as the *shortest path matrix*.

Example 3.3 Shortest path problem
Let us consider a shortest path problem for digraph shown on Fig. 3.

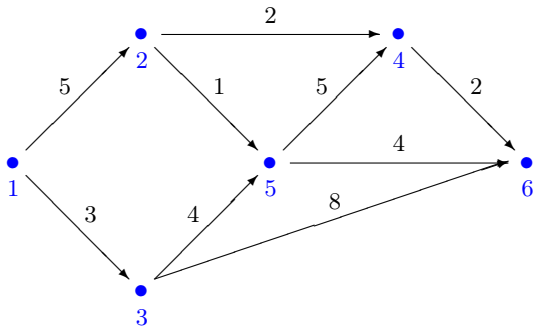

Figure 3: An exemplary digraph.

$$\mathbf{A} = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & 1 & 4 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 8 & 2 & 4 & \varepsilon \end{bmatrix},$$

so,

$$\mathbf{A}^{+} = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 7 & 2 & 9 & \varepsilon & 5 & \varepsilon \\ 6 & 1 & 4 & \varepsilon & \varepsilon & \varepsilon \\ 9 & 4 & 8 & 2 & 4 & \varepsilon \end{bmatrix}.$$

Hence, the shortest path e.g. from node 1 to node 6 is 9 — see $(\mathbf{A}^{+})_{6,1}$.

Table 8: (min, +) functions.

function	short description	
`mpm_zero`	neutral element for opertion $\vee$	
`mpm_zeros`	zeros matrix, vector or scalar	
`mpm_one`	neutral element for opertion $\wedge$	
`mpm_ones`	ones matrix (vector / scalar)	
`mpm_eye`	identity matrix	$\mathbf{I}_n$
`mpm_add`	addition	$a \vee b$, $\mathbf{A} \vee \mathbf{B}$
`mpm_multi`	multiplication	$a \wedge b$, $\mathbf{A} \wedge \mathbf{B}$
`mpm_plus(A)`	plus operator of $\mathbf{A}$	$\mathbf{A}^+ = \mathbf{A}^1 \vee \mathbf{A}^2 \vee \ldots$
`mpm_star(A)`	star operator of $\mathbf{A}$	$\mathbf{A}^\star = \mathbf{A}^0 \vee \mathbf{A}^+$
`mpm_inv(A)`	invertion of $\mathbf{A}$	$\mathbf{A}^{-1}$
`mpm_div`	division	a/b, $\mathbf{B}^{-1}\mathbf{A}$
`mpm_power`	n–th power	a^n, $\mathbf{A}^n$

4 State space description of DES

4.1 Introduction

Probably the most well–known equation in the theory of difference equations is

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1), \qquad t = 1, 2, \ldots \tag{40}$$

The vector $\mathbf{x} \in \mathbb{R}^n$ represents the *state* of the underlying model and this state evolves in time according to this equation. If an initial condition

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{41}$$

is given, then the whole future evolution of (40) is determined. Equation (40) rewritten in (max, +) notation ($\mathbf{x} \in \mathbb{R}_\varepsilon^n$, $\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$, $k \in \mathbb{N}_0$) is given by:

$$\mathbf{x}(k) = \mathbf{A} \otimes \mathbf{x}(k-1). \tag{42}$$

Where k is a cycle index.

Example 4.1 An introductory example to state space description
As an example, we take matrix $\mathbf{A} \in \mathbb{R}_\varepsilon^{2\times 2}$. Let $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$ and the initial condition $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The time evolution of (65) becomes:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \mathbf{x}(1) = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \to \mathbf{x}(2) = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \to \mathbf{x}(3) = \begin{bmatrix} 16 \\ 13 \end{bmatrix} \ldots$$

State space 1–order model with dependence on inputs and outputs is an extension of (65):

$$\begin{aligned} &\forall k \in \mathbb{N}: \\ &\quad \mathbf{x}(k) = \mathbf{A}(k)\mathbf{x}(k-1) \oplus \mathbf{B}(k)\mathbf{u}(k), &(43)\\ &\quad \mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) \oplus \mathbf{D}(k)\mathbf{u}(k). &(44) \end{aligned}$$

$\mathbf{x} \in \mathbb{R}_\varepsilon^n$ is called *state vector*,
$\mathbf{u} \in \mathbb{R}_\varepsilon^r$ is called *input vector* or *control vector* of the system,
$\mathbf{y} \in \mathbb{R}_\varepsilon^m$ is called *output vector* of the system,
$\mathbf{A} \in \mathbb{R}_\varepsilon^{n\times n}$ is called *system matrix*,
$\mathbf{B} \in \mathbb{R}_\varepsilon^{n\times r}$ is called *input matrix*,
$\mathbf{C} \in \mathbb{R}_\varepsilon^{m\times n}$ is called *output matrix*,

$\mathbf{D} \in \mathbb{R}_{\varepsilon}^{m\times r}$ is called *feedthrough (or feedforward) matrix.* In general case, for N–order time–invariant system is described by:

$$\mathbf{x}(k) = \bigoplus_{i=0}^{N} \mathbf{A}_i\mathbf{x}(k-i) \oplus \bigoplus_{i=0}^{N-1} \mathbf{B}_i\mathbf{u}(k-i), \tag{45}$$

$$\mathbf{y}(k) = \bigoplus_{i=0}^{N-1} \Big(\mathbf{C}_i\mathbf{x}(k-i) \oplus \mathbf{D}_i\mathbf{u}(k-i)\Big). \tag{46}$$

When we remove the $\mathbf{x}(k)$ on the right–side of (45) (if $\mathbf{A}_0^\star$ converge), and if we also define a new state vector, an augmented input vector and matrices:

$$\widetilde{\mathbf{x}}(k) = \big[\mathbf{x}(k)\mathbf{x}(k-1)\cdots\mathbf{x}(k-N+1)\big]^T, \tag{47}$$

$$\widetilde{\mathbf{u}}(k) = \big[\mathbf{u}(k)\mathbf{u}(k-1)\cdots\mathbf{u}(k-N+1)\big]^T, \tag{48}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0^\star\mathbf{A}_1 & \mathbf{A}_0^\star\mathbf{A}_2 & \dots & \dots & \mathbf{A}_0^\star\mathbf{A}_N \\ \mathbf{I} & \boldsymbol{\varepsilon} & \dots & \dots & \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} & & & & \\ \vdots & & \ddots & & \\ \boldsymbol{\varepsilon} & \dots & \boldsymbol{\varepsilon} & \mathbf{I} & \boldsymbol{\varepsilon} \end{bmatrix}, \tag{49}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_0^\star\mathbf{B}_0 & \dots & \mathbf{A}_0^\star\mathbf{B}_{N-1} \\ \boldsymbol{\varepsilon} & \dots & \boldsymbol{\varepsilon} \\ \vdots & & \vdots \\ \boldsymbol{\varepsilon} & \dots & \boldsymbol{\varepsilon} \end{bmatrix}, \tag{50}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_0 & \dots \mathbf{C}_{N-1} \end{bmatrix}, \tag{51}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_0 & \dots \mathbf{D}_{N-1} \end{bmatrix}, \tag{52}$$

then eqns. (45), (46) can be written as time–invariant, 1–order model of eqns. (43), (44) where $\mathbf{I}$ and $\boldsymbol{\varepsilon}$ are appropriately sized $(\max,+)$-algebraic identity and zero matrices, respectively, and

$$\forall k \in \mathbb{N} : \mathbf{A}(k) = \mathbf{A},\ \mathbf{B}(k) = \mathbf{B},\ \mathbf{C}(k) = \mathbf{C},\ \mathbf{D}(k) = \mathbf{D}.$$

Table 9: Toolbox functions for state space models.

function	short description
`mp_system`	state fo an autonomous linear max–plus system

4.2 State space description of timed event graph

Timed event graphs (TEGs) are (timed) Petri-nets [Murata 1989], which are convenient to model (timed) synchronisation problems. They are characterised by the fact that every place has exactly one predecessor transition and one successor transition. Time constraints are modelled by so–called *holding times*, representing the minimum amount of time a token has to *spend* in a place before it can contribute to enable a *downstream transition*. Let $x_i(k)$ denote the time instant, when an *internal transition* i can fire for the k^{th} time, and $\mathbf{x}(k) = (x_i(k))$ the corresponding vector of firing times. Similarly, let $u_i(k)$ denote the firing times of *input transitions* which can be triggered by the outside world, and $y_i(k)$ the firing times of *output transitions* which carry information to the outside world. It is then straightforward, to read the (max, +) equations (45)–(46) from the timed event graph.

Example 4.2 An example of a TEG and its (max, +) description
Let us consider an exemplary timed event graph depicted on Fig. 4 and its state space representaion in the (max, +) domain.

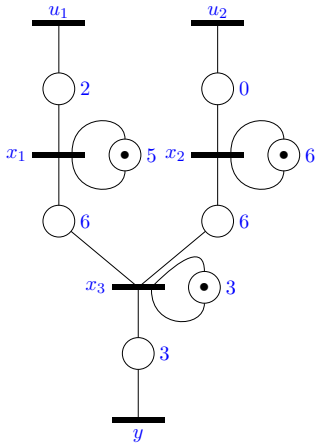

Figure 4: An exemplary TEG.

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}_0\mathbf{x}(k) \oplus \mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{B}_0\mathbf{u}(k) \\ &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k), \end{aligned}$$

where:

$$\mathbf{A}_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 6 & 6 & \varepsilon \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ \varepsilon & \varepsilon & 3 \end{bmatrix}, \mathbf{B}_0 = \begin{bmatrix} 2 \\ 0 \\ \varepsilon \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix},$$

$$\mathbf{A} = \mathbf{A}_0^{\star}\mathbf{A}_1 = \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix}, \mathbf{B} = \mathbf{A}_0^{\star}\mathbf{B}_0 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

The **Petri Net Toolbox**[2] is able to directly derive the (max, +) state space representation in form (45)–(46) from topology of a TEG.

4.3 Analysis of DES

The periodical behaviour of closed DESs, i.e. involving a set of repeatedly performed activities, can be totally characterised by solving an eigenvalue and eigenvector equation. Additionally, in the toolbox there is a set of functions for graphical illustration of DESs behaviour.

4.3.1 Graphical representation – Gantt charts

For generating Gantt charts[3] we defined two functions — see tab. 10

Table 10: Gantt charts.

function	short description
`mp_ganttr`	Gantt chart of resources occupation in time
`mp_ganttx`	Gantt chart of a state vector evolution in time

The Gantt charts can be saved as encapsulated postscript using the following expression:

```
>> print -depsc2 'fileName.eps'
```

or as a PDF:

```
>> print -dpdf 'fileName.pdf'
```

4.4 Examples of DES

In this part, we focus on two examples from domain (flexible) manufacturing.

4.4.1 A simple production system

Let consider a simple production system, presented in Fig. 6 [De Schutter 1996]. This production system consists of 3 processing units: M_1, M_2 and M_3. A raw material is fed to M_1 and M_2, where it is processed and sent to

[2] More information under http://www.ac.tuiasi.ro/pntool.

[3] At present, graphical functions work under Matlab® only.

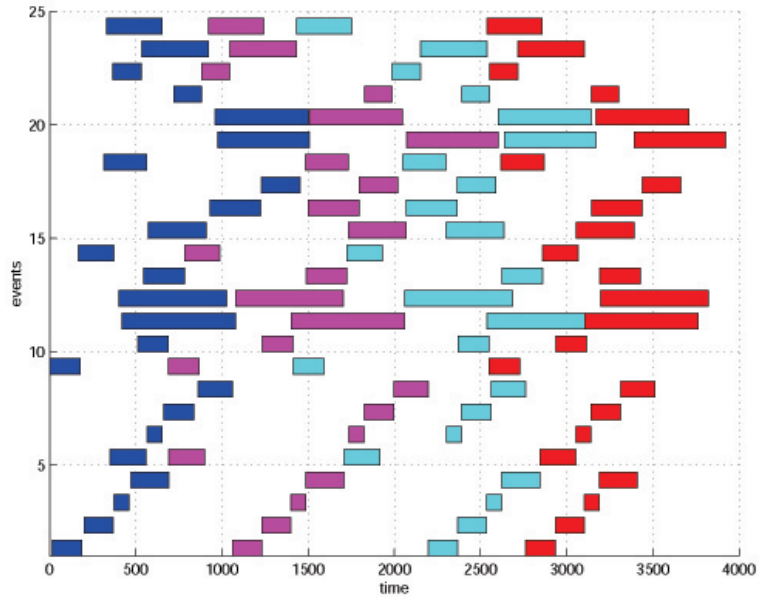

Figure 5: An exemplary Gantt chart generated by function `mp_ganttx`.

M_3, where assembly takes place. The processing times for M_1, M_2 and M_3 are respectively $d_1 = 5$, $d_2 = 6$ and $d_3 = 3$ time units. We assume that it takes $t_1 = 2$ time units for the raw material to get from the input source to M_1 and that it takes $t_3 = 1$ time unit for the finished products of processing unit M_1 to reach M_3. The other transportation times (t_2, t_4 and t_5) are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no over flow will occur. Initially all the buffers are empty and none of the processing units contains raw material or intermediate products. A processing unit can start working on a new product only after it has finished processing the previous one. We assume that each processing unit starts working as soon as all parts are available.

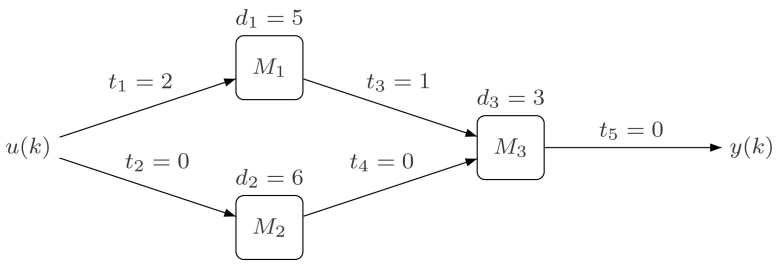

Figure 6: A simple production system.

We define:

- $u(k)$ — time instant at which raw material is feed to the system for the kth time,
- $x_i(k)$ — time instant at which the ith processing unit starts working for the kth time,
- $y(k)$ — time instant at which the kth finished product leaves the system.

Let us now determine the time instant at which processing unit M_1 starts working for the kth time. If we feed raw material to the system for the kth time, then this raw material is available at the input of processing unit M_1 at time $t = u(k) + 2$. However, M_1 can start working on the new batch of raw material only as soon as it has finished processing the previous, i.e. the $(k-1)$st batch. Since the processing time on M_1 is $d_1 = 5$ time units, the $(k-1)$st intermediate product will leave M_1 at time $t = x_1(k-1)+5$. Since M_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$\forall k \in \mathbb{N}_0 : \qquad x_1(k) = \max\big(x_1(k-1) + 5, u(k) + 2\big). \tag{53}$$

The condition that initially processing unit M_1 is empty and idle corresponds to the initial condition $x_1(0) = \varepsilon$ and hence it follows from equation (53) that $x_1(1) = u(1) + 2$, i.e. the first batch of raw material that is fed to the system will be processed immediately (after a delay of 2 time units needed to transport the raw material from the input to M_1).
Using a similar reasoning we find the following expressions for the time instants at which M_2 and M_3 start working for the kth time and for the time instant at which the kth finished product leaves the system:

$$\begin{aligned}
\forall k \in \mathbb{N}_0 : \\
x_2(k) &= \max\big(x_2(k-1) + 6, u(k) + 0\big), \\
x_3(k) &= \max\big(x_1(k) + 5 + 1, x_2(k) + 6 + 0, x_3(k-1) + 3\big) \\
&= \max\big(x_1(k-1) + 11, x_2(k-1) + 12, x_3(k-1) + 3, u(k) + 8\big), \\
y(k) &= x_3(k) + 3 + 0.
\end{aligned} \tag{54}$$

The condition that initially all buffers are empty corresponds to the initial condition

$$x_1(0) = x_2(0) = x_3(0) = \varepsilon. \tag{55}$$

Let us now rewrite the evolution equations of the production system using the (max, +) notation:

$$\begin{aligned} x_1(k) &= 5x_1(k-1) \oplus 2u(k), \\ x_2(k) &= 6x_2(k-1) \oplus u(k), \\ x_3(k) &= 11x_1(k-1) \oplus 12x_2(k-1) \oplus 3x_3(k-1) \oplus 8u(k), \\ y(k) &= 3x_3(k), \end{aligned} \tag{56}$$

in matrix notation:

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k) \\ &= \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \\ 11 & 12 & 3 \end{bmatrix} \mathbf{x}(k-1) \oplus \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \mathbf{u}(k), \end{aligned} \tag{57}$$

$$\begin{aligned} \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \\ &= \begin{bmatrix} \varepsilon & \varepsilon & 3 \end{bmatrix} \mathbf{x}(k), \end{aligned}$$

where $\mathbf{x}(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T$.

Now, we assume that after a machine has finished a sequence of products it starts with the next sequence. So, we have $\mathbf{u}(k) = \mathbf{y}(k-1)$ for $k > 0$. Hence, we obtain:

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k) \\ &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{y}(k-1) \\ &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{C}\mathbf{x}(k-1) \\ &= \big(\mathbf{A} \oplus \mathbf{B}\mathbf{C}\big)\mathbf{x}(k-1) \\ &= \widehat{\mathbf{A}}\mathbf{x}(k-1) \end{aligned} \tag{58}$$

where $\widehat{\mathbf{A}} = \mathbf{A} \oplus \mathbf{B}\mathbf{C} = \begin{bmatrix} 5 & \varepsilon & 5 \\ \varepsilon & 6 & 3 \\ 11 & 12 & 11 \end{bmatrix}$.

We can see, that this cyclic production system can also be described by a model of the form (43), but now the model is autonomous i.e. there is no external input that controls the behaviour of the system. This example is included in file: `exSimpleProduction.m`.
Exemplary results for model (57) with $\mathbf{u}(k) = 0 = constant$:

```
% definition of matrices
>> A = [      5 mp_zero mp_zero
        mp_zero       6 mp_zero
             11      12       3];
>> B = [      2       0       8]';
>> C = [mp_zero mp_zero       3];
```

```
7
8  % determine initial conditions
9  >> x0 = mp_zeros(3, 1);
10 >> u0 = 0;
11
12 % calculate a sequence of state vector
13 >> X(:, 1)= mp_add(mp_multi(A, x0), mp_multi(B, u0));
14 >> Y(1)  = mp_multi(C, X(:, 1));
15 >> for i = 2:10
16       X(:, i) = mp_add(mp_multi(A, X(:, i-1)), mp_multi(B, u0));
17       Y(i) = mp_multi(C, X(:, i));
18    end
19 >> X, Y
20 X =
21      2     7    12    17    22    27    32    37    42    47
22      0     6    12    18    24    30    36    42    48    54
23      8    13    18    24    30    36    42    48    54    60
24 Y =
25     11    16    21    27    33    39    45    51    57    63
```

`X` is a sequence of state vector from $\mathbf{x}(1)$ to $\mathbf{x}(10)$, `Y` is an output value for appropriate state vector.
Results for model (57) with $\mathbf{u}(0) = 0$ and then $\mathbf{u}(k) = \mathbf{y}(k-1)$:

```
26 % calculate a sequence of state vector
27 >> X(:, 1)= mp_add(mp_multi(A, x0), mp_multi(B, u0));
28 >> Y(1)  = mp_multi(C, X(:, 1));
29 >> for i = 2:10
30       X(:, i) = mp_add(mp_multi(A, X(:,i-1)), mp_multi(B, Y(i-1)));
31       Y(i) = mp_multi(C, X(:, i));
32    end
33 >> X, Y
34 X =
35      2    13    24    35    46    57    68    79    90   101
36      0    11    22    33    44    55    66    77    88    99
37      8    19    30    41    52    63    74    85    96   107
38 Y =
39     11    22    33    44    55    66    77    88    99   110
```

4.4.2 Multi-product manufacturing system

The idea for this example has been taken from [Baccelli et al. 1992]. Consider a manufacturing system that consists of three machines (M_1, M_2 and M_3). In this manufacturing system three different types of parts (P_1, P_2 and P_3) are produced according to a certain product mix. The routes followed by the various types of parts are depicted in Fig. 7.
Parts of type P_1 first visit machine M_2 and then go to M_3. Parts of type P_2

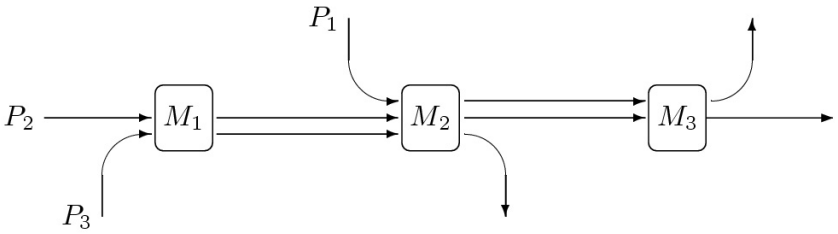

Figure 7: The routing of the various types of parts along the machines.

enter the system via machine M_1, then they go to machine M_2 and finally leave the system through machine M_3. Parts of type P_3 first visit machine M_1 and then go to M_2. It is assumed that:

- Parts are carried around on pallets. There is one pallet available for each type of part.
- It is assumed that the transportation times are negligible and that there are no set-up times on the machines when they switch from one part type to another.
- The sequencing of the various parts on the machines is known: on machine M_1 it is (P_2, P_3), i.e. the machine first processes a part of type P_2 and then a part of type P_3, on machine M_2 the sequence is (P_1, P_2, P_3), and (P_1, P_2) on machine M_3. We will call these sequences *local dispatching rules* and we will describe them as σ (i.e. σ_1 for the sequence on M_1, σ_2 for the sequence on M_2, and σ_3 for M_3).

The information about the sequencing and the duration of the various activities (processing times) is shown in Fig. 8. In this figure, the activities are represented by ordered pairs of the form (P_i, M_j) meaning that a part of type P_i is processed on machine M_j. The arcs represent the precedence constraints between activities. At the bottom right of each activity we have indicated its duration, e.g. (P_1, M_2) (activity 3) has duration $d_3 = 3$.

In order to simplify the process of deriving the evolution equations of this system, we shall first look at what happens in one cycle of the production process. We define:

- $u_i(k)$ — time instant at which machine M_i is available for the first activity that should be performed on it in the kth production cycle for $i = 1, 2, 3$;
- $u_j(k)$ — time instant at which the raw material for a part of type P_{j-3} is available in the kth production cycle for $j = 4, 5, 6$;
- $x_i(k)$ — time instant at which activity i starts in the kth production cycle for $i = 1, 2, \ldots, 7$;

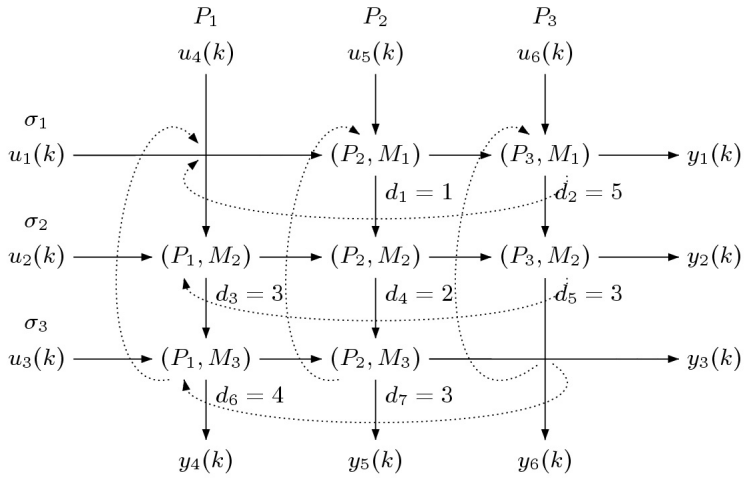

Figure 8: The sequence and the duration of the various activities.

- $y_i(k)$ — time instant at which machine M_i has finished processing the last part of the kth production cycle that should be processed on it for $i = 1, 2, 3$;
- $y_j(k)$ — time instant at which the finished product of type P_{j-3} of the kth production cycle has been completed for $j = 4, 5, 6$.

We have the following evolution equations:

$$\begin{aligned} x_1(k) &= 5x_2(k-1) \oplus 3x_7(k-1) \oplus u_1(k) \oplus u_5(k), \\ x_2(k) &= 1x_1(k) \oplus 3x_5(k-1) \oplus u_6(k), \\ &\vdots \end{aligned} \tag{59}$$

or, more compactly:

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}_0\mathbf{x}(k) \oplus \mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{B}_0\mathbf{u}(k), \\ &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k), \end{aligned} \tag{60}$$

where $\mathbf{A} = \mathbf{A}_0^*\mathbf{A}_1$ and $\mathbf{B} = \mathbf{A}_0^*\mathbf{B}_0$.

In the cyclic (closed) systems, it is advisable to introduce a matrix $\mathbf{K} \in \mathbb{R}_\varepsilon^{r\times m}$, to describe the dynamics of restarting the system for the next cycle:

$$\mathbf{u}(k) = \mathbf{K}\mathbf{y}(k-1). \tag{61}$$

Hence:

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{K}\mathbf{y}(k-1), && (62)\\ &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{K}\mathbf{C}\mathbf{x}(k-1), && (63)\\ &= (\mathbf{A} \oplus \mathbf{B}\mathbf{K}\mathbf{C})\mathbf{x}(k-1), && (64)\\ &= \hat{\mathbf{A}}\mathbf{x}(k-1). && (65) \end{aligned}$$

So, we obtain an autonomous model.

In the steady–state, the difference in the same state variables of two subsequent cycles is (in the conventional algebra):

$$\mathbf{x}(k) = \mathbf{x}(k-1) + T, \tag{66}$$

or in (max, +):

$$\mathbf{x}(k) = T\mathbf{x}(k-1), \tag{67}$$

where T is a cycle time.
Hence, from (65) and (67):

$$\mathbf{A}\mathbf{x}(k-1) = T\mathbf{x}(k-1), \tag{68}$$

where T is a (max, +) eigenvalue of $\mathbf{A}$.

An exemplary analysis of the considered system based on the Max–Plus Algebra Toolbox is shown below. It provides cycle time and consecutive states ($\mathbf{x}(k)$) of the system. It is assumed that $\mathbf{x}(0) = [\varepsilon]^{7\times 1}$, i.e. all machines are idle in the beginning, and $\mathbf{u}(1) = [e]^{6\times 1}$, i.e. all machines can be started without delay. This example is included in files `exMultiProduct.m` and `exGanttr.m`.

```
% operation times
>> d = [1 5 3 2 3 4 3];

% matrices definition
>> A0 = mp_zeros(7);
>> A0(2,1) = d(1); A0(4,1) = d(1); A0(4,3) = d(3); A0(5,2) = d(2);
>> A0(5,4) = d(4); A0(6,3) = d(3); A0(7,4) = d(4); A0(7,6) = d(6);

>> A1 = mp_zeros(7);
>> A1(1,2) = d(2); A1(1,7) = d(7); A1(2,5) = d(5);
>> A1(3,5) = d(5); A1(3,6) = d(6); A1(6,7) = d(7);

>> B0 = mp_zeros(7,6);
>> B0(1,1) = mp_one; B0(1,5) = mp_one; B0(2,6) = mp_one;
>> B0(3,2) = mp_one; B0(3,4) = mp_one; B0(6,3) = mp_one;

```

```
>> C = mp_zeros(6,7);
>> C(1,2) = d(2); C(2,3) = d(3); C(3,7) = d(7);
>> C(4,6) = d(6); C(5,7) = d(7); C(6,5) = d(5);

>> K = mp_eye(6);

% create the matrices A=(A0^*A1) and B=(A0^*B0)
>> A = mp_multi(mp_star(A0), A1);
>> B = mp_multi(mp_star(A0), B0);

% create matrix M = (A \oplus BKC)
>> M = mp_add(A, mp_multi(B, mp_multi(K, C)));

% determine initial conditions
x0 = mp_zeros(7, 1);
u1 = mp_ones(6, 1);

% calculate a sequence of a state vector
>> X(:, 1) = mp_add(mp_multi(A, x0), mp_multi(B, u1));
>> for i = 2:10
       X(:, i) = mp_multi(M, X(:, i-1));
   end
>> X
X =
     0    10    19    29    38    48    57    67    76    86
     1    11    20    30    39    49    58    68    77    87
     0     9    19    28    38    47    57    66    76    85
     3    12    22    31    41    50    60    69    79    88
     6    16    25    35    44    54    63    73    82    92
     3    12    22    31    41    50    60    69    79    88
     7    16    26    35    45    54    64    73    83    92

% calculate a sequence of an output vector
>> Y = mp_multi(C, X)
Y =
     6    16    25    35    44    54    63    73    82    92
     9    19    28    38    47    57    66    76    85    95
    10    19    29    38    48    57    67    76    86    95
     7    16    26    35    45    54    64    73    83    92
    10    19    29    38    48    57    67    76    86    95
     9    19    28    38    47    57    66    76    85    95

% cycle time
>> lambda = mp_mcm(M)
lambda = 9.5000

% does the system start in steady-state?
>> mp_isegv(M, X(:,1), lambda)
ans = 0
```

```
66
67  % this means NO, so, let us calculate new x0 for start in steady state
68  >> x0 = mp_egv1(M, mp_ones(7,1))
69  x0 =
70     14.5000
71     15.5000
72     14.0000
73     17.0000
74     20.5000
75     17.0000
76     21.0000
77
78  % let min(x0)==0
79  >> x0 = x0 - min(x0);
80
81  % calculate a new sequence of a state vector
82  >> X(:, 1) = x0;
83  >> for i = 2:10
84         X(:, i) = mp_multi(M, X(:, i-1));
85     end
86
87  >> X
88  X =
89     0.50  10.00  19.50  29.00  38.50  48.00  57.50  67.00  76.50  86.00
90     1.50  11.00  20.50  30.00  39.50  49.00  58.50  68.00  77.50  87.00
91        0   9.50  19.00  28.50  38.00  47.50  57.00  66.50  76.00  85.50
92     3.00  12.50  22.00  31.50  41.00  50.50  60.00  69.50  79.00  88.50
93     6.50  16.00  25.50  35.00  44.50  54.00  63.50  73.00  82.50  92.00
94     3.00  12.50  22.00  31.50  41.00  50.50  60.00  69.50  79.00  88.50
95     7.00  16.50  26.00  35.50  45.00  54.50  64.00  73.50  83.00  92.50
96
97  % calculate others preformance indices
98  % resources utilisation level (for M1, M2, M3 respectively):
99  >> ro = [d(1)+d(2) d(3)+d(4)+d(5) d(6)+d(7)] / lambda
100 ro =
101     0.6316    0.8421    0.7368
102
103 % processes execution level (P1, P2, P3):
104 >> eta = [d(3)+d(6) d(1)+d(4)+d(7) d(2)+d(5)] / lambda
105 eta =
106     0.7368    0.6316    0.8421
```

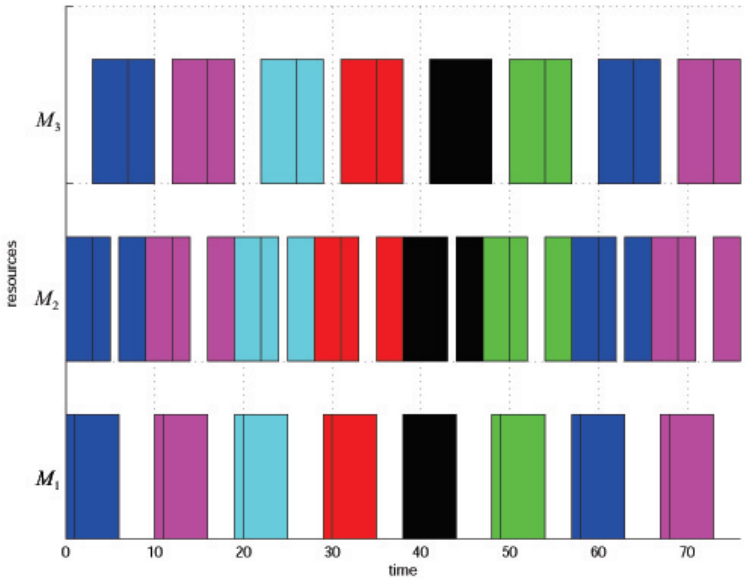

Figure 9: A usage of machines in time, the Gantt chart generated by function `mp_ganttr`.

5 Miscellaneous functions and data structures

5.1 Functions

Table 11: Miscellaneous functions.

function	short description
mp_mxconv	matrix conversion from the (max, +) to the (min, +) and vice versa
mp_mx2latex	matrix (or vector) conversion from the (max, +) to the LaTeX
mpm_mx2latex	matrix (or vector) conversion from the (min, +) to the LaTeX

5.2 Data structures

Definition 5.1 *resources–state–vector matrix* ***MPX_Rsv***
Let us define a matrix MPX_Rsv of resources–state–vector connection, $\texttt{MPX\_Rsv} \in \mathbb{N}_0^{p\times q}$, where p is a number of resources in the system, q is a maximal amount of the state–vector entries, which describe one resource, i.e. the i–th row define i–th resource, entries of this row are numbers of entries of the state–vector which are described this resource.

Example 5.2
Let us consider an example from § 4.4.2.
There are 3 resources M_1, M_2 and M_3. Resource M_1 is described by states x_1, x_2, M_2 by x_3, x_4, x_5, and M_3 by x_6, x_7. So

$$\texttt{MPX\_Rsv} = \begin{bmatrix} M_1 & = & \{x_1, & x_2\} & \\ M_2 & = & \{x_3, & x_4, & x_5\} \\ M_3 & = & \{x_6, & x_7\} & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix}.$$

Rows 1 and 3 have 2 elements, the row 2 has 3 elements, thus we have to fill missing element by 0 to obtain the same length of all rows (add zeros in rows 1 and 3).
See function mp_ganttr or example exGanttr.m for details.

6 Toolbox function reference

All functions described below are listed in alphabetical order.

---

mp_add

(max, +) addition of scalars, vectors or matrices

Syntax

```
y = mp_add(n, m)
```

Description

$y = n \oplus m$

- If `n` and `m` are scalars, result is a (max, +) sum of `n` and `m`.
- If `n` (or `m`) is scalar and `m` (or `n`) is vector, result is a vector the same size as `m` (or `n`) where for every entries is (max, +) added `n` (or `m`).
- If `n` (or `m`) is scalar and `m` (or `n`) is matrix, result is a matrix the same size as `m` (or `n`) where for every entries is (max, +) added `n` (or `m`).
- If `n` and `m` are this same size matrices, result is an array the same size as `n` (and `m`) with the entries equal to (max, +) addition elements from `n` and `m`.

Example

```
>> mp_add(-5, 7)
ans =
        7

>> mp_add(3, [mp_zero 4])
ans =
        3     4

>> A = [1 6 8 ;  3 -Inf 4], B = [2 5 -Inf ; 3 1 3]
A =
        1        6        8
        3     -Inf        4

```

```
B =
        2        5     -Inf
        3        1        3

>> mp_add(A, B)
ans =
        2        6        8
        3        1        4
```

See also

mp_multi, mp_one, mp_ones, mp_zero, mp_zeros

---

mp_conv

scalar, vector or matrix conversion from the (max, +) to the (min, +) and vice versa

Syntax

```
Z = mp_conv(X)
```

Description

Function exchanges all ∞ and $-\infty$ to $-\infty$ and ∞ respectively. Entries with other values are not changing.

Example

```
>> A = [0        3      Inf        1
        1        2        2     -Inf
     -Inf      Inf        1        0];

>> mp_conv(A)
ans =
        0        3     -Inf        1
        1        2        2      Inf
      Inf     -Inf        1        0
```

---

mp_div

(max, +) division

Syntax

```
Z = mp_div(A, B)
[Z, err] = mp_div(A, B)
```

Description

$Z = A \oslash B$

- If `A` is a scalar
  - if `B` is a scalar, result is a scalar: (max, +) division of `A` by `B`;
  - if `B` is a vector (or matrix), result is a vector (or a matrix) where every entries are: `A` divided by apropriate entry of `B`.
- If `A` is a vector (or a matrix)
  - if `B` is a scalar, result is a vector (or a matrix) the same size as `A` where every entries of `A` are (max, +) divided by `B`;
  - if `B` is a vector (or a matrix) the same size as `A`: the result is (max, +) division `A` by `B`, i.e. `mp_multi(mp_inv(B), A)`, and when `B` is (max, +)-invertable `err = 0`, otherwise `err = 1`;
  - if `B` is a vector (or a matrix) different size than `A` — operation is not defined.
- Division by (max, +) zero (`-Inf`) is not defined - and returns `NaN`.

Example

```
>> mp_div(3, mp_one)
ans =
        3

>> mp_div(3, 3)
ans =
        0

>> mp_div(3, mp_zero)
ans =
      NaN

>> mp_div(3, [mp_zero 7 mp_one])
ans =
      NaN       -4        3
```

```
>> mp_div([mp_zero 7 mp_one], 3)
ans =
     -Inf        4       -3
```

(max, +) division for square matrices:

```
>> A = [mp_zero  1        mp_zero;
        2        mp_zero  mp_zero;
        mp_zero  mp_zero  3];
>> B = [3        mp_zero  mp_zero;
        mp_zero  mp_zero  4;
        mp_zero  5  mp_zero];
>> [C, err] = mp_div(A, B)
C =
     -Inf       -2     -Inf
     -Inf     -Inf       -2
       -2     -Inf     -Inf

err =
        0

>> mp_multi(B, C)
ans =
     -Inf        1     -Inf
        2     -Inf     -Inf
     -Inf     -Inf        3
```

and not square matrices:

```
>> A = [mp_zero  1  mp_zero;  2  mp_zero  mp_zero];
>> B = [3  mp_zero  mp_zero;  mp_zero  mp_zero  4];
>> [C, err] = mp_div(A, B)
C =
     -Inf       -2     -Inf
     -Inf     -Inf     -Inf
       -2     -Inf     -Inf

err =
        0

>> mp_multi(B, C)
ans =
     -Inf        1     -Inf
        2     -Inf     -Inf
```

See also

mp_inv, mp_multi, mp_one, mp_zero

---

mp_egv_bo93

eigenvector and eigenvalue of matrix A

Syntax

```
eigenvector = mp_egv_bo93(A, x0)
eigenvector = mp_egv_bo93(A, x0, r)
[eigenvector, eigenvalue] = mp_egv_bo93(A, x0)
[eigenvector, eigenvalue] = mp_egv_bo93(A, x0, r)
```

Description

Function returns an eigenvector and an eigenvalue of matrix A by [Braker and G.-J. Olsder 1993], see alg. 2, page 23.

- A must be a square matrix, $A \in \mathbb{R}_{\varepsilon}^{n \times n}$
- algorithm starts from vector x0 $\in \mathbb{R}^n$
- r (optional) is the maximum number of steps, after which the algorithm stops — default r = 1000
- if A has more than one eigenvalue, function returns only one
- if A has more than one eigenvector associated with the eigenvalue, function returns only one

Example

Let us consider example 2.43.

```
>> A  = [mp_zero        3  mp_zero        1;
                2  mp_zero        1  mp_zero;
                1        2        2  mp_zero;
          mp_zero  mp_zero        1  mp_zero];

>> x0 = [      0  mp_zero  mp_zero  mp_zero]';

>> [v, l] = mp_egv_bo93(A, x0)
v =
  10.0000
```

```
   9.5000
   9.0000
   7.5000

l =
   2.5000
```

See also

mp_egv_o91, mp_egv_pqc, mp_egv_sw001, mp_egv_sw002, mp_is_egv1, mp_is_egv2, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_egv_o91

(candidate) eigenvector and eigenvalue of matrix `A`

Syntax

```
eigenvector = mp_egv_o91(A, x0)
eigenvector = mp_egv_o91(A, x0, r)
[eigenvector, eigenvalue] = mp_egv_o91(A, x0)
[eigenvector, eigenvalue] = mp_egv_o91(A, x0, r)
```

Description

Function returns a (candidate) `eigenvector` and an `eigenvalue` of matrix `A` by [G.-J. Olsder 1991] — alg. 1, page 23.

- `A` must be a square matrix, $\mathtt{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$,
- algorithm starts from vector $\mathtt{x0} \in \mathbb{R}^n$,
- `r` (optional) is the maximum number of steps, after which the algorithm stops, default $\mathtt{r} = 1000$
- If `A` has more than one eigenvalue, function returns only one.
- If `A` has more than one eigenvector associated with the `eigenvalue`, function returns only one.

Example

Let us consider example 2.42.

```
>> A = [3 7; 2 4], x0 = [0 0]'
A =
```

```
         3         7
         2         4

x0 =
         0
         0

>> [v, l] = mp_egv_o91(A, x0)
v =
   9.0000
   6.5000

l =
   4.5000
```

See also

mp_egv_bo93, mp_egv_pqc, mp_egv_sw001, mp_egv_sw002, mp_is_egv1, mp_is_egv2, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_egv_pqc

eigenvalue of A calculated form components p, q and c

Syntax

```
l = mp_egv_pqc(p, q, c)
l = mp_egv_pqc([p q c])
```

Description

- Function calculates an eigenvalue of matrix A from components p,q and c (see theorem 2.40).
- The components can be calculated by mp_pqc.
- $p,q \in \mathbb{N}_0 : p > q \geqslant 0$, $c \in \mathbb{R}$

Example

Let us consider example 2.41.

```
>> A = [3 7; 2 4]
A =
```

```
         3         7
         2         4

>> x0 = [0 0]'
x =
         0
         0

>> mp_egv_pqc(mp_pqc(A, x0))
ans =
   4.5000
```

See also

mp_pqc, mp_egv_bo93, mp_egv_o91, mp_egv_sw001, mp_egv_sw002, mp_is_egv1, mp_is_egv2, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_egv_sw001

eigenvector and eigenvalue of matrix `A`

Syntax

```
eigenvector = mp_egv_sw001(A, x0)
eigenvector = mp_egv_sw001(A, x0, r)
[eigenvector, eigenvalue] = mp_egv_sw001(A, x0)
[eigenvector, eigenvalue] = mp_egv_sw001(A, x0, r)
```

Description

Function returns an `eigenvector` and an `eigenvalue` of matrix `A` by [Subiono and van der Woude 2000], see alg. 4, page 28.

- `A` must be a square matrix, $\mathtt{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$,
- algorithm starts from vector $\mathtt{x0} \in \mathbb{R}^n$,
- `r` (optional) is the maximum number of steps, after which the algorithm stops, default $\mathtt{r} = 1000$.
- If `A` has more than one eigenvalue, function returns only one.
- If `A` has more than one eigenvector associated with the `eigenvalue`, function returns only one.

Example

Let us consider example 2.46.

```
>> A = [3 7; 2 4], x0 = [0 0]'
A =
        3        7
        2        4

x0 =
        0
        0

>> [v, l] = mp_egv_sw001(A, x0)
v =
  11.5000
   9.0000

l =
   4.5000
```

See also

mp_egv_bo93, mp_egv_o91, mp_egv_pqc, mp_egv_sw002, mp_is_egv1, mp_is_egv2, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_egv_sw002

eigenvector and eigenvalue of matrix `A`

Syntax

```
eigenvector = mp_egv_sw002(A, x0)
eigenvector = mp_egv_sw002(A, x0, r)
[eigenvector, eigenvalue] = mp_egv_sw002(A, x0)
[eigenvector, eigenvalue] = mp_egv_sw002(A, x0, r)
```

Description

Function returns an `eigenvector` and an `eigenvalue` of matrix `A` by [Subiono and van der Woude 2000], see alg. 5, page **??**.

- `A` must be a square matrix, $\texttt{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$,
- algorithm starts from vector `x0` $\in \mathbb{R}^n$,
- `r` (optional) is the maximum number of steps, after which the algorithm stops, default `r` $= 1000$.
- If `A` has more than one eigenvalue, function returns only one.
- If `A` has more than one eigenvector associated with the `eigenvalue`, function returns only one.

Example

Let us consider example 2.45.

```
>> A  = [mp_zero        3  mp_zero        1;
               2  mp_zero        1  mp_zero;
               1        2        2  mp_zero;
         mp_zero  mp_zero        1  mp_zero];

>> x0 = [      0  mp_zero  mp_zero  mp_zero]';

>> [v, l] = mp_egv_sw002(A, x0)
v =
  12.5000
  12.0000
  11.5000
  10.0000

l =
   2.5000
```

See also

mp_egv_bo93, mp_egv_o91, mp_egv_pqc, mp_egv_sw001, mp_is_egv1, mp_is_egv2, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_ev_fw

eigenvectors of matrix `A`

Syntax

```
[eigenvectors] = mp_eg_fw(A)
```

Description

Function returns a set of eigenvectors of `A` from final Floyd–Warshall matrix (genenerated by `mp_mcm_fw` or `mp_mx_fw`), without recurrent eigenvectors, as well as without their linear combinations. Function uses the Floyd–Warshall algorithm Ahuja, Magnanti, and Orlin 1993 adapted by G.-J. Olsder, Roos, and van Egmond 1999 — see step 3 of alg. 5, page 25.

Example

Let us consider example 2.44.

```
>> A=[      4  mp_zero  mp_zero  mp_zero  mp_zero        4;
            2        5        1  mp_zero  mp_zero  mp_zero;
      mp_zero  mp_zero  mp_zero        6  mp_zero  mp_zero;
      mp_zero  mp_zero        8        3  mp_zero  mp_zero;
            9  mp_zero  mp_zero  mp_zero  mp_zero  mp_zero;
      mp_zero  mp_zero  mp_zero  mp_zero        8
         3];

>> F = mp_mx_fw(A, 7)
F =
        0     -Inf     -Inf     -Inf       -2       -3
       -5       -2       -6       -7       -7       -8
     -Inf     -Inf        0       -1     -Inf     -Inf
     -Inf     -Inf        1        0     -Inf     -Inf
        2     -Inf     -Inf     -Inf        0       -1
        3     -Inf     -Inf     -Inf        1        0

>> ev = mp_ev_fw(F)
ev =
        0        0       -2       -3       -2
       -5       -5       -6       -6       -7
        0       -1        0        0       -1
        1        0        1        1        0
        2        2        0       -1        0
        3        3        1        0        1
```

See also

`mp_mcm_fw`, `mp_mx_fw`

mp_eye

(max, +) identity matrix

Syntax

```
Y = mp_eye
Y = mp_eye(n)
Y = mp_eye(n, m)
```

Description

- `mp_eye` returns 0.
- `mp_eye(n)` or `mp_eye([n])` returns an `n`–by–`n` (max, +) identity matrix, i.e. with (max, +) units on the main diagonal and ε elsewhere.
- `mp_eye(n, m)` or `mp_eye([n m])` returns an `n`–by–`m` (max, +) identity matrix.

Example

```
>> mp_eye
ans =
        0

>> mp_eye(2)
ans =
        0     -Inf
     -Inf        0

>> mp_eye(2, 3)
ans =
        0     -Inf     -Inf
     -Inf        0     -Inf
```

See also

mp_one, mp_ones, mp_zero, mp_zeros, mp_randi

---

mp_ganttr

Gantt chart of resources occupation in time

Syntax

```
mp_ganttr(X, time, MPX_Rsv)
mp_ganttr(X, time, MPX_Rsv, xrange)
mp_ganttr(X, time, MPX_Rsv, ytick)
mp_ganttr(X, time, MPX_Rsv, xrange, ytick)
```

Description

A Gantt chart of resources occupation in time, where

`X`	a $n \times m$ matrix of state vectors, i.e. a collection of m successive state vectors
`time`	a $n \times 1$ vector of operation times for every entry in state vector (time of every event/operation), or a $n \times m$ matrix of operation times for m iterations (time of every event/operation in every iteration)
`MPX_Rsv`	a matrix of resources–state–vector connection for details see def. 5.1
`xrange`	(optional) enables to specify limits of the x axis, `xrange = [xmin xmax]`, by default, function finds the maximum and minimum of the data i.e. `xmin = min(X)`, `xmax = max(X+t)`
`ytick`	`ytick = 0|1` (optional, default `ytick = 1`) if `ytick = 1` then every ytick is marked along y axis

Example

```
% collections of state vectors
>> V = [0   10   19   29   38   48   57   67   76   86;
        1   11   20   30   39   49   58   68   77   87;
        0    9   19   28   38   47   57   66   76   85;
        3   12   22   31   41   50   60   69   79   88;
        6   16   25   35   44   54   63   73   82   92;
        3   12   22   31   41   50   60   69   79   88;
        7   16   26   35   45   54   64   73   83   92];

% operation times
>> op_times = [1 5 3 2 3 4 3]';

% matrix of resources–state–vector connection
>> R_V = [[1 2 0]
          [3 4 5]
          [6 7 0]];

>> mp_ganttr(X(:, 1:8), t, R);
```

A Gantt chart generated by the code presented above is shown in fig. 9.

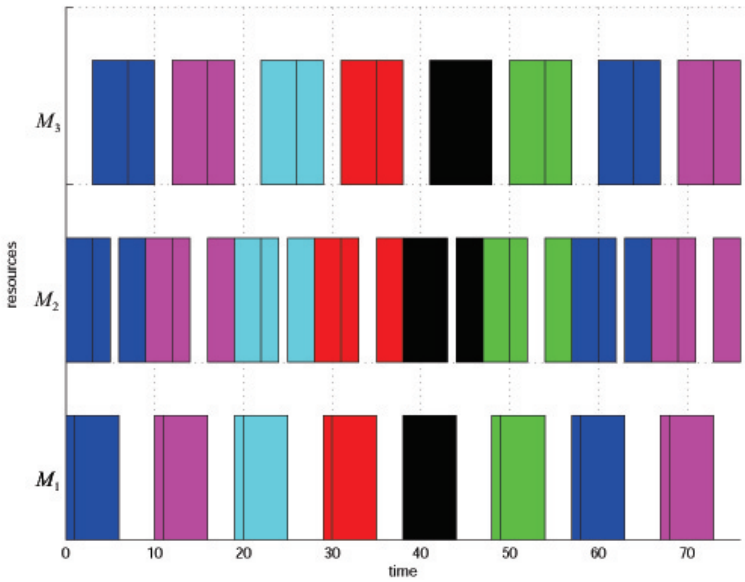

Figure 10: A usage of resources in time, the Gantt chart generated by function `mp_ganttr`.

See also

`mp_ganttx`

---

mp_ganttx

Gantt chart of a state vector evolution in time
Gantt chart of execution of operations

Syntax

```
mp_ganttx(X, time)
mp_ganttx(X, time, xrange)
mp_ganttx(X, time, ytick)
mp_ganttx(X, time, xrange, ytick)
```

Description

A Gantt chart of a state vector evolution in time (or execution of operations)

`X`	a $n \times m$ matrix of state vectors, i.e. a collection of m successive state vectors
`t`	a $n \times 1$ vector of operation times for every entry in state vector (time of every event/operation), or a $n \times m$ matrix of operation times for m iterations (time of every event/operation in every iteration)
`xrange`	(optional) enables to specify limits of the x axis, `xrange = [xmin xmax]`, by default, function finds the maximum and minimum of the data i.e. `xmin = min(X)`, `xmax = max(X+t)`
`ytick`	`ytick = 0|1` (optional, default `ytick = 1`) if `ytick = 1` then every ytick is marked along y axis

Example

A part of code from file `exGanttx.m` demonstrates use of `mp_ganttx`:

```
1  % collections of 4 state vectors
2  V = [    14        1060        2196        2762
3         201        1232        2368        2934
4         374        1400        2536        3102
```

⋮

```
25        331         921        1432        2537];
26
27 operation_times = [172;
28                    168;
29                     87;
```

⋮

```
50                    321];
51
52 mp_ganttx(V, operation_times, [0 4000], 0);
```

A Gantt chart generated by the code presented above is shown in fig. 11.

See also

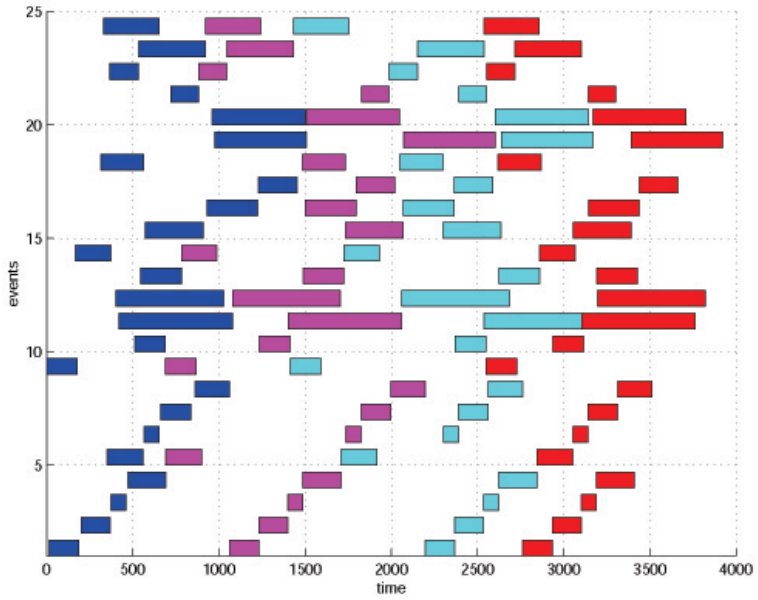

Figure 11: An exemplary Gantt chart generated by function `mp_ganttx`.

mp_ganttr

---

mp_inv

(max, +) matrix invertion

Syntax

```
Y = mp_inv(A)
[Y, err] = mp_inv(A)
```

Description

`Y` = $\mathtt{A}^{-1}$

- If `A` is a scalar then `Y = mp_power(A, -1)` and `err = 0`.
- If `A` is a square matrix then `Y` = $\mathtt{A}^{-1}$ and if `AY = YA = I` then `err = 0`, otherwise `err = 1`.
- If `A` not a square matrix then `Y = (I-A)'` and if `AY = I` then `err = 0`, otherwise `err = 1`.

Example

Let us consider a square matrix:

```
>> mp_inv(5)
ans =
       -5

>> A = [mp_zero  1        mp_zero;
        2        mp_zero  mp_zero;
        mp_zero  mp_zero  3];

>> [Y, err] = mp_inv(A)
Y =
     -Inf       -2     -Inf
       -1     -Inf     -Inf
     -Inf     -Inf       -3

err =
        0

>> mp_multi(A, Y)
ans =
        0     -Inf     -Inf
     -Inf        0     -Inf
     -Inf     -Inf        0
```

And now not a square matrix:

```
>> A = [mp_zero  1        mp_zero;
        2        mp_zero  mp_zero];

>> [Y, err] = mp_inv(A)
Y =
     -Inf       -2
       -1     -Inf
     -Inf     -Inf

err =
        0

>> mp_multi(A, Y)
ans =
        0     -Inf
     -Inf        0
```

See also

mp_power, mp_multi, mp_div, mp_eye

---

mp_is_egv1

checks, if vector x is an eigenvector of A

Syntax

```
y = mp_is_egv1(A, x)
y = mp_is_egv1(A, x, eigenvalue)
y = mp_is_egv1(A, x, eigenvalue, d)
```

Description

Function returns 1 if x is an eigenvector of A, otherwise 0.

- A must be a square matrix, $\mathtt{A} \in \mathbb{R}_{\varepsilon}^{n \times n}$,
- $\mathtt{x} \in \mathbb{R}^{n}$,
- eigenvalue of A (optional), $\mathtt{eigenvalue} \in \mathbb{R}$, default it is computed by mp_mcm(A),
- d (optional) a number of arcs in the maximum cycle mean of $\mathcal{G}(\mathtt{A})$, default $\mathtt{d} = 1$,
- if $\mathtt{d} \in \mathbb{N}$ is given, the function checks eq. (69) and returns 1 if (69) is fulfiled, 0 otherwise.

$$\mathtt{A}^{\mathtt{d}}\mathtt{x} = \mathtt{eigenvalue}^{\mathtt{d}}\mathtt{x}, \tag{69}$$

Example

```
>> A = [2 5; 3 3]; x = [1 0]';
>> mp_is_egv1(A, x)
ans =
        1

>> % let's check:
>> mp_multi(A, x)
ans =
        5
        4

>> mp_multi(mp_mcm(A), x)
ans =
        5
        4
```

See also

mp_egv_bo93, mp_egv_o91, mp_egv_pqc, mp_egv_sw001, mp_egv_sw002, mp_is_egv2, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_is_egv2

checks, if vector x is an eigenvector of A

Syntax

```
y = mp_is_egv1(x2, x1, eigenvalue)
y = mp_is_egv1(x2, x1, eigenvalue, d)
```

Description

Function returns 1 if eigenvalue $\otimes$ x1 = x2, 0 otherwise.

- x1, x2 $\in \mathbb{R}^n$, eigenvalue $\in \mathbb{R}$,
- if d $\in \mathbb{N}$ is given (optional, default d = 1) then
  - function returns 1 if eigenvalue$^{\text{d}}$ $\otimes$ x1 = x2, 0 otherwise.

Example

```
>> A = [2 5; 3 3]
A =
        2        5
        3        3
>> lambda = mp_mcm(A)
lambda =
        4

>> x = [1 0]'
x =
        1
        0

>> mp_multi(A, x)
ans =
        5
        4
```

```
18
19  >> mp_is_egv2(ans, x, lambda)
20  ans =
21          1
```

See also

mp_egv_bo93, mp_egv_o91, mp_egv_pqc, mp_egv_sw001, mp_egv_sw002, mp_is_egv1, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_is_pga

is a precedence graph $\mathcal{G}(\mathtt{A})$ acyclic?
has a matrix `A` got any eigenvalue?

Syntax

```
y = mp_is_pga(A)
```

Description

- Function returns 1:
  - if $\mathcal{G}(\mathtt{A})$ is acyclic ($\mathcal{G}(\mathtt{A})$ does not contain any circuit), or equivalently
  - if `A` has NOT got any eigenvalue,
- 0 otherwise.
- `A` must be a square matrix.

Example

Let us consider an acyclic digraph from Fig. 1, matrix `A` represents this graph.

```
1  >> A = [mp_zero        4  mp_zero;
2                2  mp_zero  mp_zero;
3          mp_zero        3        2];
4
5  >> mp_is_pga(A)
6  ans =
7          0
```

See also

mp_is_pgc, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_karp, mp_mcm_fw, mp_isegv, mp_isegvv, mp_compute_pqc, mp_lpqc, mp_FloydWarshallMx, mp_egvFloydWarshall, mp_egv1, mp_egv2, mp_egv3, mp_egv4

---

mp_is_pgc

is precedence graph $\mathcal{G}(\mathtt{A})$ connected?

Syntax

```
y = mp_is_pgc(A)
```

Description

- Function returns 1 if precedence graph $\mathcal{G}(\mathtt{A})$ is connected, 0 otherwise.
- `A` must be a square matrix.

Example

Let us consider a digraph from Fig. 1.

```
>> A = [mp_zero        4  mp_zero;
              2  mp_zero  mp_zero;
       mp_zero        3        2];

>> mp_is_pgc(A)
ans =
        1
```

See also

mp_is_pga, mp_is_pgsc1, mp_is_pgsc2

---

mp_is_pgsc1

is precedence graph $\mathcal{G}(\mathtt{A})$ strongly connected?
is matrix `A` irreducible?
has matrix `A` got exactly one eigenvalue?

Syntax

```
y = mp_is_pgsc1(A)
```

Description

- Function returns 1:
  - if $\mathcal{G}(\mathtt{A})$ is strongly connected, or equivalently
  - if matrix `A` is irreducible, or
  - if matrix `A` has got exactly one eigenvalue,
- 0 otherwise.
- `A` must be a square matrix.
- Result is calculated based directly on definition 2.27

Example

Let us consider a digraph from Fig. 1.

```
>> A = [mp_zero        4  mp_zero;
              2  mp_zero  mp_zero;
       mp_zero        3        2];

>> mp_is_pgsc1(A)
ans =
       0
```

See also

mp_is_pga, mp_is_pgc, mp_is_pgsc2, mp_mcm, mp_mcm_karp, mp_mcm_fw, mp_isegv, mp_isegvv, mp_compute_pqc, mp_lpqc, mp_FloydWarshallMx, mp_egvFloydWarshall, mp_egv1, mp_egv2, mp_egv3, mp_egv4

---

mp_is_pgsc2

is precedence graph $\mathcal{G}(\mathtt{A})$ strongly connected?
is matrix `A` irreducible?
has matrix `A` got exactly one eigenvalue?

Syntax

```
y = mp_is_pgsc2(A)
```

Description

Like mp_is_pgsc1, function checks whether $\mathcal{G}(\mathtt{A})$ is strongly connected (from theorem 2.29).

- Function returns 1:
  - if $\mathcal{G}(\mathtt{A})$ is strongly connected, or equivalently
  - if matrix `A` is irreducible, or
  - if matrix `A` has got exactly one eigenvalue,
- 0 otherwise.
- `A` must be a square matrix.

See also

mp_is_pga, mp_is_pgc, mp_is_pgsc1, mp_mcm, mp_mcm_karp, mp_mcm_fw, mp_isegv, mp_isegvv, mp_compute_pqc, mp_lpqc, mp_FloydWarshallMx, mp_egvFloydWarshall, mp_egv1, mp_egv2, mp_egv3, mp_egv4

---

mp_mcm

(max, +) eigenvalue of matrix `A`
maximum cycle mean of precedence graph $\mathcal{G}(\mathtt{A})$

Syntax

```
l = mp_mcm(A)
[l, d] = mp_mcm(A)
```

Description

- (max, +) eigenvalue of an irreducible matrix `A`
  - if `A` is not irreducible, it gives only one eigenvalue.
- Maximum cycle mean of precedence graph $\mathcal{G}(\mathtt{A})$
  - if $\mathcal{G}(\mathtt{A})$ is not strongly connected, it gives only one value.
- Function returns:
  - `l` — the eigenvalue of `A` or equivalently, the value of maximum cycle mean of precedence graph $\mathcal{G}(\mathtt{A})$, from (16)
  - `d` — the number of arcs in critical circuit.
- `A` must be a square matrix.

Example

Let us consider the digraph from Fig. 2.

```
>> A = [      5  mp_zero        5;
        mp_zero        6        3;
             11       12       11];

>> [l, d] = mp_mcm(A)
l =
       11
d =
        1
```

See also

mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm_fw, mp_mcm_karp, mp_isegv, mp_isegvv, mp_compute_pqc, mp_lpqc, mp_FloydWarshallMx, mp_egvFloydWarshall, mp_egv1, mp_egv2, mp_egv3, mp_egv4

---

mp_mcm_fw

(max, +) eigenvalue of matrix `A`
maximum cycle mean of precedence graph $\mathcal{G}$(`A`)
Floyd–Warshall's algorithm

Syntax

```
eigenvalue = mp_mcm_fw(A)
[eigenvalue, Afw] = mp_mcm_fw(A)
```

Description

- (max, +) eigenvalue of a matrix `A`
  - if `A` is not irreducible, it gives only one eigenvalue.
- Maximum cycle mean of precedence graph $\mathcal{G}$(`A`).
  - if $\mathcal{G}$(`A`) is not strongly connected, it gives only one value.
- Function returns:
  - `eigenvalue` — an eigenvalue of `A` (a maximum cycle mean of $\mathcal{G}$(`A`)),
  - `Afw` — a final Floyd–Warshall's matrix,
  - from `Afw` matrix can be obtained a set of eigenvectors by mp_egvFloydWarshall.
- `A` must be a square matrix.

Function uses the Floyd–Warshall procedure [Ahuja, Magnanti, and Orlin 1993] adapted by [G.-J. Olsder, Roos, and van Egmond 1999] — see steps 1–2 of alg. 5, page 25.

Example

Let us consider example 2.44

```
>> A=[4        mp_zero  mp_zero  mp_zero  mp_zero  4;
      2        5        1        mp_zero  mp_zero  mp_zero;
      mp_zero  mp_zero  mp_zero  6        mp_zero  mp_zero;
      mp_zero  mp_zero  8        3        mp_zero  mp_zero;
      9        mp_zero  mp_zero  mp_zero  mp_zero  mp_zero;
      mp_zero  mp_zero  mp_zero  mp_zero  8        3];

>> l = mp_mcm_fw(A)
l =
        7

>> v = [0 -5 0 1 2 3]';

>> mp_multi(A, v)' == mp_multi(l, v)'
ans =
        1        1        1        1        1        1
```

See also

mp_ev_fw, mp_mx_fw, mp_egv_bo93, mp_egv_o91, mp_egv_pqc, mp_egv_sw001, mp_egv_sw002, mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_is_egv1, mp_is_egv2, mp_mcm, mp_mcm_karp

---

mp_mcm_karp

(max, +) eigenvalue of matrix A
maximum cycle mean of precedence graph $\mathcal{G}$(A)
Karp's algorithm

Syntax

```
l = mp_mcm_karp(A)
```

Description

- (max, +) eigenvalue of an irreducible matrix A

  - if A is not irreducible, it gives only one eigenvalue.
- Maximum cycle mean of precedence graph $\mathcal{G}(\mathtt{A})$
  - if $\mathcal{G}(\mathtt{A})$ is not strongly connected, it gives only one value.
- Function returns:
  - the eigenvalue of A, or equivalently
  - the value of maximum cycle mean of precedence graph $\mathcal{G}(\mathtt{A})$.
- A must be a square matrix.

Karp's algorithm [Karp 1978] adapted by [Gaubert and Scilab 1998], see page 18.

Example

Let us consider the matrix **A** from Fig. 2.

```
>> A = [      5  mp_zero       5;
        mp_zero        6       3;
             11       12      11];

>> mp_mcm_karp(A)
ans =
       11
```

See also

mp_is_pga, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_isegv, mp_isegvv, mp_compute_pqc, mp_lpqc, mp_FloydWarshallMx, mp_egvFloydWarshall, mp_egv1, mp_egv2, mp_egv3, mp_egv4

---

mp_multi

(max, +) multiplication of scalars, vectors or matrices

Syntax

```
y = mp_multi(m, n)
```

Description

$\mathtt{y} = \mathtt{n} \otimes \mathtt{m}$

- If n and m are scalars, result is a (max, +) multiplication of n and m.

- If `n` (or `m`) is scalar and `m` (or `n`) is vector, result is a vector the same size as `m` (or `n`) where for every entries is (max, +) multiplication by `n` (or `m`).
- If `n` (or `m`) is scalar and `m` (or `n`) is matrix, result is a matrix the same size as `m` (or `n`) where for every entries is (max, +) multiplication by `n` (or `m`).
- If `n` is an $p \times q$ matrix and `m` is an $q \times r$ matrix result is a (max, +) product `n` and `m` returns an $p \times r$ matrix.

Example

```
>> mp_multi(-5, 7)
ans =
        2

>> mp_multi(3, [mp_zero 4])
ans =
     -Inf        7

>> A=[1 6 2; 8 3 4],  B=[2 5; 3 3; 1 6]
A =
        1        6        2
        8        3        4

B =
        2        5
        3        3
        1        6

>> mp_multi(A, B)
ans =
        9        9
       10       13
```

See also

mp_add, mp_div, mp_inv, mp_power, mp_one, mp_ones, mp_zero, mp_zeros

---

mp_mx_fw

final Floyd–Warshall's matrix

Syntax

`F = mp_mx_fw(A, eigenvalue)`

Description

Floyd–Warshall algorithm [Ahuja, Magnanti, and Orlin 1993] adapted by [G.-J. Olsder, Roos, and van Egmond 1999] — see alg. 3, page 25 for details.

- Function returns a final Floyd–Warshall matrix `F` from square matrix `A` and its `eigenvalue`.
- From this matrix can be obtained set of eigenvectors of `A` by `mp_ev_fw`.

Example

Let us consider example 2.44.

```
>> A=[      4  mp_zero  mp_zero  mp_zero  mp_zero        4;
            2        5        1  mp_zero  mp_zero  mp_zero;
      mp_zero  mp_zero  mp_zero        6  mp_zero  mp_zero;
      mp_zero  mp_zero        8        3  mp_zero  mp_zero;
            9  mp_zero  mp_zero  mp_zero  mp_zero  mp_zero;
      mp_zero  mp_zero  mp_zero  mp_zero        8
         3];

>> F = mp_mx_fw(A, 7)
F =
        0     -Inf     -Inf     -Inf       -2       -3
       -5       -2       -6       -7       -7       -8
     -Inf     -Inf        0       -1     -Inf     -Inf
     -Inf     -Inf        1        0     -Inf     -Inf
        2     -Inf     -Inf     -Inf        0       -1
        3     -Inf     -Inf     -Inf        1        0

>> ev = mp_ev_fw(F)
ev =
        0        0       -2       -3       -2
       -5       -5       -6       -6       -7
        0       -1        0        0       -1
        1        0        1        1        0
        2        2        0       -1        0
```

```
24          3        3        1        0        1
```

See also

mp_ev_fw, mp_mcm_fw

---

mp_mx2latex

matrix conversion from the Matlab (max, +) description to the LaTeX

Syntax

```
mp_mx2latex(X)
mp_mx2latex(X, 'fileName')
```

Description

(max, +) matrix (or vector) conversion from the Matlab®/Octave notation to the LaTeX source code.

- `X` is a matrix (or vector) to conversion;
- `fileName` (optional) is a name of file in which a LaTeX source code of `X` will be saved, default it is `mp_mx.tex`.

Example

```
>> A = [ 0     3   Inf     1
         1     2     2  -Inf
      -Inf   Inf     1     0];

>> mp_mx2latex(A, 'A.tex')
```

LaTeX source of A.tex:

```
\left[
\begin{array}{*{20}c}
0 & 3 & \infty & 1 \\
1 & 2 & 2 & \varepsilon \\
\varepsilon & \infty & 1 & 0 \\
\end{array}
\right]
```

LaTeX result:

$$\begin{bmatrix} 0 & 3 & \infty & 1 \\ 1 & 2 & 2 & \varepsilon \\ \varepsilon & \infty & 1 & 0 \end{bmatrix}$$

See also

mp_mx2latex, mp_conv

---

mp_one

(max, +) unit (0), neutral element for oprtation $\otimes$

Syntax

```
y = mp_one
```

Description

`mp_one` returns 0.

Example

```
>> mp_one
ans =
        0
```

See also

mp_ones, mp_zero, mp_zeros, mp_eye, mp_multi

---

mp_ones

(max, +) ones (0's) matrix, vector or scalar

Syntax

```
y = mp_ones
y = mp_ones(n)
y = mp_ones(n, m)
```

Description

- `mp_ones` returns 0.
- `mp_ones(n)` or `mp_ones([n])` is an `n`–by–`n` matrix of 0–s.
- `mp_ones(n, m)` or `mp_ones([n, m])` is an `n`–by–`m` matrix of 0–s.

Example

```
>> mp_ones
ans =
        0
>> mp_ones(2)
ans =
        0        0
        0        0

>> mp_ones(2, 3)
ans =
        0        0        0
        0        0        0
```

See also

mp_one, mp_zero, mp_zeros, mp_eye, mp_multi

---

mp_power

(max, +) raising to a power

Syntax

```
y = mp_power(x, n)
Y = mp_power(X, n)
```

Description

(max, +) `n`–th power of `x`, $y = x^n$
(max, +) `n`–th power of square matrix `X`, $Y = X^n$

- If `x` is a scalar than from definition 2.2:
  - $x \in \mathbb{R}_\varepsilon, n \in \mathbb{R} : x^n = x \times n$ (in conventional algebra).
  - If $n = 0$ then $x^0 = 0$.
  - If $x = \varepsilon$ and $n > 0$ then $\varepsilon^n = \varepsilon$.
  - If $x = \varepsilon$ and $n < 0$ then ε^n is not defined.

  - If `x` $= \varepsilon$ and `n` < 0 then $\varepsilon^0 = 0$ by definition.
- If `n` is a vector (or a matrix) then result is a vector (or a matrix) this same size, where every entries is (max, $+$) power of `x` to the proper element from `n`.
- If `X` is a square matrix then `n` must belong to $\mathbb{N}_0 \cup \{-1\}$, see definition 2.15.
  - If `n` $\notin \mathbb{N}_0 \cup \{-1\}$ operation is not defined.
- If `X` is not a scalar nor square matrix operation is not defined.

Example

```
>> mp_power(3, 3)
ans =
        9
>> mp_power(3, -2)
ans =
       -6

>> mp_power(3, 1/4)
ans =
   0.7500

>> mp_power(2, [1/4 1 4 -4])
ans =
  0.50000  2.00000  8.00000  -8.00000

>> mp_power([1 6; -Inf 3], 3)
ans =
        3       12
     -Inf        9
```

See also

mp_multi, mp_inv, mp_div, mp_eye, mp_one, mp_zero

---

mp_pqc

computes components of an eigenvalue for mp_egv_pqc

Syntax

```
[p q c] = mp_pqc(A, x0)
[p q c] = mp_pqc(A, x0, r)
```

Description

- Function calculates the components `p, q, c` of an eigenvalue of matrix `A` (see theorem 2.40).
- Algorithm starts from vector `x0`.
- `r` (optional, default `r = 1000`) is the maximum number of steps, after which the algorithm stops.
- The eigenvalue of `A` from the results of `mp_pqc` can be calculated by `mp_egv_pqc`.
- `A` must be a square matrix.

Example

Let us consider example 2.41.

```
>> A = [3 7; 2 4]
A =
        3        7
        2        4

>> x0 = [0 0]'
x =
        0
        0

>> mp_pqc(A, x0)
ans =
        3        1        9

>> mp_egv_pqc(ans)
ans =
   4.5000
```

See also

`mp_egv_bo93`, `mp_egv_o91`, `mp_egv_pqc`, `mp_egv_sw001`, `mp_egv_sw002`, `mp_is_egv1`, `mp_is_egv2`, `mp_is_pga`, `mp_is_pgsc1`, `mp_is_pgsc2`, `mp_mcm`, `mp_mcm_fw`, `mp_mcm_karp`

mp_randi

(max, +) random integer and ε

Syntax

```
Y = mp_randi(IMAX)
Y = mp_randi(IMAX, N)
Y = mp_randi(IMAX, M, N)
Y = mp_randi([IMIN IMAX], ...)
```

Description

- `mp_randi` returns a random integers in the range 1:IMAX (or IMIN:IMAX) + `mp_zero`.
- Additional arguments determine the shape of the return matrix `Y`.
- For more info look at `randi`.

Example

```
>> mp_randi([0 10])
ans =
        2

>> mp_randi([0 10], 3, 3)
ans =
        0        1        8
        3     -Inf        4
        0       10        9
```

See also

mp_ones, mp_zeros

---

mp_solve_Axb

greatest subsolution of `Ax = b`
(max, +) residuation operation

Syntax

```
x = mp_solvAxb(A, b)
[x, err] = mp_solvAxb(A, b)
```

Description

(max, +) residuation

- The greatest subsolution of `Ax` = `b`, computes the largest `x` such that `Ax` $\preccurlyeq$ `b` (see theorem 2.32).
- For `A` and `b` finite scalars, `x` = `A` − `b` (in conventional algebra).
- If `x` is the solution of `Ax` = `b` then `err = 0`, otherwise `err = 1`.

Example

```
% first example
>> a = 1; b = 2;
>> x = mp_solve_Axb(a, b)
x =
        1

% let's check
>> mp_multi(a, x)
ans =
        2

% second example
>> A = [2 3; 4 5]
A =
        2        3
        4        5

>> b = [6 8]'
b =
        6
        8

>> [x, err] = mp_solve_Axb(A, b)
x =
        4
        3

err =
        0

>> mp_multi(A, x)
ans =
```

```
        6
        8
```

As we can see, this result is the solution of $\mathbf{Ax} = \mathbf{b}$, but depends on parameters, a result can be a subsolution only, e.g.

```
% example with subsolution
>> b = [6 7]'
b =
        6
        7

>> [x, err] = mp_solve_Axb(A, b)
x =
        3
        2

err =
        1

>> mp_multi(A, x)
ans =
        5
        7
```

See also

mp_star, mp_solve_xAxb

---

mp_solve_xAxb

(max, +) solution of $\mathtt{x} = \mathtt{Ax} \oplus \mathtt{b}$

Syntax

```
x = mp_solve_xAxb(A, b)
```

Description

- It solves $\mathbf{x} = \mathtt{Ax} \oplus \mathtt{b}$ in the (max, +) algebra, i.e. $\mathtt{x} = \mathtt{A}^{\star}\mathtt{b}$.
- When there is no circuits with positive weight in the precedence graph $\mathcal{G}(\mathtt{A})$, then $\mathtt{x} = (\mathtt{A}^0 \oplus \mathtt{A}^1 \oplus \cdots \oplus \mathtt{A}^{(m-1)}) \otimes \mathtt{b}$, where m denotes the order of the square matrix A.

- `A` must be a square matrix.

Example

```
>> x = mp_solve_xAxb(−5, 2)
x =
        2

>> x == mp_add(p_multi(−5, x), 2)
ans =
        1

>> A = [mp_zero 2 3; −2 −10 −1; −5 −2 mp_one]
A =
     −Inf        2        3
       −2      −10       −1
       −5       −2        0

>> b = [1 2 3]'
b =
        1
        2
        3

>> x = mp_solve_xAxb(A, b)
x =
        6
        4
        3

>> x == mp_add(mp_multi(A, x), b)
ans =
        1
        1
        1
```

See also

mp_star, mp_solve_Axb

---

mp_star

(max, +) star operator

Syntax

```
B = mp_star(A)
[B, n] = mp_star(A)
```

Description

$\mathtt{B} = \mathtt{A}^\star = \mathtt{A}^0 \oplus \mathtt{A}^1 \oplus \ldots$

- It solves $\mathbf{x} = \mathtt{A}\mathbf{x} \oplus \mathbf{b}$ in the (max, +) algebra (i.e. $\mathbf{x} = \mathtt{A}^\star\mathbf{b}$).
- When there is no circuits with positive weight in the precedence graph $\mathcal{G}(\mathtt{A})$, then $\mathtt{A}^\star = \mathtt{A}^0 \oplus \mathtt{A}^1 \oplus \cdots \oplus \mathtt{A}^{(m-1)}$, where m denotes the order of the square matrix `A`.
- Function returns:
  - `star` $= \mathtt{A}^\star$
  - `n` — a minimal value for what all entries in $\mathtt{A}^{\mathtt{n}}$ are equal to ε.
- `A` must be a square matrix.

Example

```
>> mp_star(2)
ans =
      Inf
>> mp_star(-1)
ans =
        0

>> mp_star(mp_zero)
ans =
        0

>> mp_star(mp_one)
ans =
        0

>> mp_star(mp_zeros(2, 2))
ans =
        0     -Inf
     -Inf        0

>> [B, n] = mp_star([mp_zero mp_zero; 6 mp_zero])
B =
```

```
        0     -Inf
        6        0

n =
        2

>> A = [mp_zero 2 3; -2 -10 -1; -5 -2 mp_one]
A =
     -Inf        2        3
       -2      -10       -1
       -5       -2        0

>> [B, n] = mp_star(A)
B =
        0        2        3
       -2        0        1
       -4       -2        0

n =
      Inf

>> [B, n] = mp_star([2 3; mp_zero -1])
B =
      Inf      Inf
     -Inf        0

n =
      Inf
```

Lets find the minimal solution of the eq. (25):

```
>> a = -1; b = 2;
>> x = mp_multi(mp_star(a), b)
x =
        2

>> mp_add(mp_multi(a, x), b) == x
ans =
        1
```

for matrices:

```
>> A = [mp_zero   mp_one    mp_zero;
        mp_zero   mp_zero        -1;
        mp_one    mp_zero   mp_zero];
```

```
>> b = [     10;  mp_zero;  mp_zero];
>> x = mp_multi(mp_star(A), b)
x =
       10
        9
       10
```

Is it correct? Let us compare results from eqns. (25) and (26):

```
>> mp_add(mp_multi(A, x), b) == x
ans =
        1
        1
        1
```

See also

mp_solve_xAxb, mp_solve_Axb, mp_is_pga, mp_is_pgc, mp_is_pgsc1, mp_is_pgsc2, mp_mcm, mp_mcm_fw, mp_mcm_karp

---

mp_system

state fo an autonomous linear max–plus system

Syntax

```
X = mp_system(A, x0, k)
X = mp_system(A, x0, k, one_only)
```

Description

States of an autonomous linear max–plus system

$$x(k) = Ax(k-1),$$

where:

`A`	a (max, +) system matrix;
`k`	a cycle index (iteration number);
`x0`	an initial state vector, `x0 = x(0)`;
`one_only`	(optional, default `one_only = 1`) if is set to 1, the result is a state vector `x(k)`, if `one_only = 0`, the result is a matrix of a collection of state vectors from `x(0)` to `x(k)`.

Example

```
>> A = [3 7; 2 4]
A =
        3        7
        2        4

>> x0 = [1 0]'
x0 =
        1
        0

>> mp_system(A, x0, 8)
ans =
       25
       22

>> mp_system(A, x0, 5, 0)
ans =
        1        7       11       16       20       25
        0        4        9       13       18       22
```

See also

mp_multi

---

mp_trace

(max, +) trace of a matrix

Syntax

```
y = mp_trace(A)
```

Description

the (max, +) trace of matrix `A`, i.e.
the (max, +) sum of main diagonal of `A` (or equivalently a maximal element from main diagonal of `A`).

Example

```
>> A = [5 mp_zero 5; mp_zero 6 3; 11 12 11]
A =
        5     -Inf        5
```

```
    -Inf     6     3
      11    12    11

>> mp_trace(A)
ans =
      11
```

See also

mp_add

---

mp_zero

(max, +) zero ($-\infty$), i.e. ε, neutral element for operation $\oplus$

Syntax

```
y = mp_zero
```

Description

`mp_zero` returns `-Inf`

Example

```
>> mp_zero
ans =
    -Inf
```

See also

mp_zeros, mp_one, mp_ones, mp_eye, mp_add

---

mp_zeros

(max, +) zeros matrix, vector or scalar

Syntax

```
y = mp_zeros
y = mp_zeros(n)
y = mp_zeros(n, m)
```

Description

- `mp_zeros` returns ε, i.e. `-Inf`.
- `mp_zeros(n)` or `mp_zeros([n])` returns an `n`–by–`n` matrix of ε–s.
- `mp_zeros(n, m)` or `mp_zeros([n m])` returns an `n`–by–`m` matrix of ε–s.

Example

```
>> mp_zeros
ans =
     -Inf

>> mp_zeros(2)
ans =
     -Inf     -Inf
     -Inf     -Inf

>> mp_zeros(2, 3)
ans =
     -Inf     -Inf     -Inf
     -Inf     -Inf     -Inf
```

See also

mp_zero, mp_one, mp_ones, mp_eye, mp_add

---

mpm_add

(min, +) addition of scalars, vectors or matrices

Syntax

`y = mpm_sum(n,m)`

Description

`y` = `n` $\vee$ `m`

- If `n` and `m` are scalars, result is a (min, +) sum of `n` and `m`.
- If `n` (or `m`) is scalar and `m` (or `n`) is vector, result is a vector the same size as `m` (or `n`) where for every entries is (min, +) added `n` (or `m`).

- If `n` (or `m`) is scalar and `m` (or `n`) is matrix, result is a matrix the same size as `m` (or `n`) where for every entries is (min, +) added `n` (or `m`).
- If `n` and `m` are this same size matrices, result is an array the same size as `n` (and `m`) with the entries equal to (min, +) addition elements from `n` and `m`.

Example

```
>> mpm_add(-5, 7)
ans =
      -5

>> mpm_add(3, [mpm_zero 4])
ans =
        3        3
>> A = [1 6; 8 3], B = [2 5; 3 3]
A =
        1        6
        8        3

B =
        2        5
        3        3

>> ans =

        1        5
        3        3
```

See also

mpm_multi, mpm_one, mpm_ones, mpm_zero, mpm_zeros

---

mpm_div

(min, +) division

Syntax

```
Z = mpm_div(A, B)
[Z, err] = mpm_div(A, B)
```

Description

`Z = A ⊘ B`

- If `A` is a scalar
  - if `B` is a scalar, result is a scalar: (min, +) division of `A` by `B`;
  - if `B` is a vector (or matrix), result is a vector (or a matrix) where every entries are: `A` divided by apropriate entry of `B`.
- If `A` is a vector (or a matrix)
  - if `B` is a scalar, result is a vector (or a matrix) the same size as `A` where every entries of `A` are (min, +) divided by `B`;
  - if `B` is a vector (or a matrix) the same size as `A`: the result is (min, +) division `A` by `B`, i.e. `mpm_multi(mpm_inv(B), A)`, and when `B` is (min, +)-invertable `err = 0`, otherwise `err = 1`;
  - if `B` is a vector (or a matrix) different size than `A` — operation is not defined.
- Division by (min, +) zero (`Inf`) is not defined - and returns `NaN`.

Example

```
>> mpm_div(3, 3)
ans =
        0

>> mpm_div(3, mpm_one)
ans =
        3

>> mpm_div(3, mpm_zero)
ans =
      NaN

>> mpm_div([mpm_zero 7 mpm_one], 3)
ans =
      Inf        4       -3

>> mpm_div(3, [mpm_zero 7 mpm_one])
ans =
      NaN       -4        3

>> A = [mpm_zero         1  mpm_zero;
```

```
               2  mpm_zero  mpm_zero;
       mpm_zero  mpm_zero         3];

>> B = [       3  mpm_zero  mpm_zero;
       mpm_zero  mpm_zero         4;
       mpm_zero         5  mpm_zero];

>> [C, err] = mpm_div(A, B)
C =
      Inf       -2      Inf
      Inf      Inf       -2
       -2      Inf      Inf

err = 0

>> mpm_multi(B, C)
ans =
      Inf        1      Inf
        2      Inf      Inf
      Inf      Inf        3
```

See also

mpm_inv, mpm_multi, mpm_one, mpm_zero

---

mpm_eye

(min, +) identity matrix

Syntax

```
Y = mpm_eye
Y = mpm_eye(n)
Y = mpm_eye(n, m)
```

Description

- `mpm_eye` returns 0.
- `mpm_eye(n)` or `mpm_eye([n])` returns an `n`–by–`n` (min, +) identity matrix with 0–s on the main diagonal and `Inf`'s elsewhere.
- `mpm_eye(n, m)` or `mpm_eye([n m])` returns an `n`–by–`m` (min, +) identity matrix.

Example

```
>> mpm_eye
ans =
        0

>> mpm_eye(2)
ans =
        0      Inf
      Inf        0

>> mpm_eye(2, 3)
ans =
        0      Inf      Inf
      Inf        0      Inf
```

See also

mpm_one, mpm_ones, mpm_zero, mpm_zeros, mpm_randi

---

mpm_inv

(min, +) matrix invertion

Syntax

```
Y = mpm_inv(A)
[Y, err] = mpm_inv(A)
```

Description

$Y = A^{-1}$

- If A is a scalar then Y = mpm_power(A, -1) and err = 0.
- If A is a square matrix then $Y = A^{-1}$ and if AY = YA = I then err = 0, otherwise err = 1.
- If A is not a square matrix then Y = (I-A)' and if AY = I then err = 0, otherwise err = 1.

Example

Let us consider a square matrix:

```
>> mpm_inv(5)
ans =
       -5

>> A = [mpm_zero         1  mpm_zero;
               2  mpm_zero  mpm_zero;
        mpm_zero  mpm_zero         3];

>> [Y, err] = mpm_inv(A)
Y =
      Inf       -2      Inf
       -1      Inf      Inf
      Inf      Inf       -3

err =
        0

>> A = [mpm_zero         1  mpm_zero;
               2  mpm_zero  mpm_zero];

>> [Y, err] = mpm_inv(A)
Y =
      Inf       -2
       -1      Inf
      Inf      Inf
err =
        0

>> mpm_multi(A, Y)
ans =
        0      Inf
      Inf        0
```

See also

mpm_power, mpm_multi, mpm_div, mpm_eye

---

mpm_multi

(min, +) multiplication of scalars, vectors or matrices

Syntax

```
y = mpm_multi(m, n)
```

Description

$\mathtt{y} = \mathtt{n} \wedge \mathtt{m}$

- If `n` and `m` are scalars, result is a (min, +) multiplication of n and m.
- If `n` (or `m`) is scalar and `m` (or `n`) is vector, result is a vector the same size as `m` (or `n`) where for every entries is (min, +) multiplication by `n` (or `m`).
- If `n` (or `m`) is scalar and `m` (or `n`) is matrix, result is a matrix the same size as `m` (or `n`) where for every entries is (min, +) multiplication by `n` (or `m`).
- If `n` is an $p \times q$ matrix and `m` is an $q \times r$ matrix result is a (min, +) product `n` and `m` returns an $p \times r$ matrix.

Example

```
>> mpm_multi(-5, 7)
ans =
        2

>> mpm_multi(3, [mpm_zero 4])
ans =
      Inf        7

>> A = [1 6 2; 8 3 4],  B = [2 5; 3 3; 1 6]
A =
        1        6        2
        8        3        4

B =
        2        5
        3        3
        1        6

mpm_multi(A, B)
ans =
        3        6
        5        6
```

See also

mpm_add, mpm_div, mpm_inv, mpm_power, mpm_one, mpm_ones, mpm_zero, mpm_zeros

---

mpm_mx2latex

matrix conversion from the Matlab (min, +) description to the LaTeX

Syntax

```
mpm_mx2latex(X)
mpm_mx2latex(X, 'fileName')
```

Description

(min, +) matrix (or vector) conversion from the Matlab®/Octave notation to the LaTeX source code.

- `X` is a matrix (or vector) to conversion;
- `fileName` (optional) is a name of file in which a LaTeX source code of `X` will be saved, default it is `mpm_mx.tex`.

Example

```
>> A = [ 0     3   Inf     1
         1     2     2  -Inf
      -Inf   Inf     1     0];

>> mpm_mx2latex(A, 'A.tex')
```

LaTeX source of A.tex:

```
\left[
\begin{array}{*{20}c}
0 & 3 & \varepsilon & 1 \\
1 & 2 & 2 & -\infty \\
-\infty & \varepsilon & 1 & 0 \\
\end{array}
\right]
```

LaTeX result:

$$\left[\begin{array}{*{20}c} 0 & 3 & \varepsilon & 1 \\ 1 & 2 & 2 & -\infty \\ -\infty & \varepsilon & 1 & 0 \\ \end{array}\right]$$

See also

mpm_mx2latex, mp_conv

---

mpm_one

(min, +) unit (0), neutral element for oprtation $\wedge$

Syntax

```
y = mpm_one
```

Description

`mpm_one` returns 0.

Example

```
>> mpm_one
ans =
        0
```

See also

mpm_ones, mpm_zero, mpm_zeros, mpm_eye, mpm_multi

---

mpm_ones

(min, +) ones (0's) matrix, vector or scalar

Syntax

```
y = mpm_ones
y = mpm_ones(n)
y = mpm_ones(n,m)
```

Description

- `mpm_ones` returns 0.
- `mpm_ones(n)` or `mpm_ones([n])` is an `n`–by–`n` matrix of 0–s.
- `mpm_ones(n,m)` or `mpm_ones([n,m])` is an `n`–by–`m` matrix of 0–s.

Example

```
>> mpm_ones
ans =
         0

>> mpm_ones(2)
ans =
        0        0
        0        0

>> mpm_ones(2, 3)
ans =
        0        0        0
        0        0        0
```

See also

mpm_one, mpm_zero, mpm_zeros, mpm_eye, mpm_multi

---

mpm_plus

(min, +) plus operator
shortest path matrix

Syntax

```
Y = mpm_plus(A)
[Y, n] = mpm_plus(A)
```

Description

$$\mathtt{Y} \quad = \quad \mathtt{A}^{+} = \mathtt{A}^{1} \vee \mathtt{A}^{2} \vee \ldots$$
$$\mathtt{A}^{+} \vee \mathtt{A}^{0} = \mathtt{A}^{\star}$$

When there is no circuits with negative weight in the precedence graph $\mathcal{G}(\mathtt{A})$, then

$$\mathtt{A}^{+} = \mathtt{A}^{1} \vee \mathtt{A}^{2} \vee \cdots \vee \mathtt{A}^{(m-1)}$$

where m denotes the order of the square matrix A.

Function returns:

- `Y` = A^{+}
- `n` — a minimal value for what all entries in A^{n} are equal to ε.

`A` must be a square matrix.

Example

```
>> mpm_plus(2)
ans =
        2

>> mpm_add(mpm_one, mpm_plus(2)) == mpm_star(2)
ans =
        1
```

Let us consider a shortest path problem — example 3.3.

```
>>A=[mpm_zero mpm_zero mpm_zero mpm_zero mpm_zero mpm_zero;
     5        mpm_zero mpm_zero mpm_zero mpm_zero mpm_zero;
     3        mpm_zero mpm_zero mpm_zero mpm_zero mpm_zero;
     mpm_zero 2        mpm_zero mpm_zero 5        mpm_zero;
     mpm_zero 1        4        mpm_zero mpm_zero mpm_zero;
     mpm_zero mpm_zero 8        2        4        mpm_zero]

>> mpm_plus(A)
ans =
      Inf      Inf      Inf      Inf      Inf      Inf
        5      Inf      Inf      Inf      Inf      Inf
        3      Inf      Inf      Inf      Inf      Inf
        7        2        9      Inf        5      Inf
        6        1        4      Inf      Inf      Inf
        9        4        8        2        4      Inf
```

See also

mpm_star, mp_is_pga, mp_is_pgc

---

mpm_power

(min, +) raising to a power of scalar or a square matrix

Syntax

```
y = mpm_power(x, n)
Y = mpm_power(X, n)
```

Description

(min, +) `n`–th power of `x`, $\mathtt{y} = \mathtt{x}^n$
(min, +) `n`–th power of square matrix `X`, $\mathtt{Y} = \mathtt{X}^n$

- If `x` is a scalar than:
  - $\mathtt{x} \in \mathbb{R}_\varepsilon, \mathtt{n} \in \mathbb{R} : \mathtt{x}^n = \mathtt{x} \times \mathtt{n}$ (in conventional algebra).
  - If $\mathtt{n} = 0$ then $x^0 = 0$.
  - If $\mathtt{x} = \varepsilon$ and $\mathtt{n} > 0$ then $\varepsilon^n = \varepsilon$.
  - If $\mathtt{x} = \varepsilon$ and $\mathtt{n} < 0$ then ε^n is not defined.
  - If $\mathtt{x} = \varepsilon$ and $\mathtt{n} < 0$ then $\varepsilon^0 = 0$ by definition.
- If `n` is a vector (or a matrix) then result is a vector (or a matrix) this same size, where every entries is (min, +) power of `x` to the proper element from `n`.
- If `X` is a square matrix then `n` must belong to $\mathbb{N}_0 \cup \{-1\}$,
  - If $\mathtt{n} \notin \mathbb{N}_0 \cup \{-1\}$ operation is not defined.
- If `X` is not a scalar nor square matrix operation is not defined.

Example

```
>> mpm_power(3, 3)
ans =
        9

>> mpm_power(3, -2)
ans =
       -6

>> mpm_power(3, 1/4)
ans =
   0.7500

>> mpm_power(2, [1/4 1 4 -4])
ans =
   0.5000   2.0000   8.0000  -8.0000

>> mpm_power([1 6; 8 3], 3)
ans =
        3        8
       10        9
```

See also

mpm_multi, mpm_inv, mpm_div, mpm_eye, mpm_one, mpm_zero

---

mpm_star

(min, +) star operator

Syntax

```
B = mpm_star(A)
[B, n] = mpm_star(A)
```

Description

$\mathtt{B} = \mathtt{A}^{\star} = \mathtt{A}^{0} \vee \mathtt{A}^{+} = \mathtt{A}^{0} \vee \mathtt{A}^{1} \vee \dots$ It solves $\mathbf{x} = \mathtt{A}\mathbf{x} \vee \mathbf{b}$ in the (min, +) algebra (i.e. $\mathbf{x} = \mathtt{A}^{\star}\mathbf{b}$).
When there is no circuits with negative weight in the precedence graph $\mathcal{G}(\mathtt{A})$, then

$$\mathtt{A}^{\star} = \mathtt{A}^{0} \vee \mathtt{A}^{1} \vee \cdots \vee \mathtt{A}^{(m-1)}$$

where m denotes the order of the square matrix A.

Function returns:

- $\mathtt{B} = \mathtt{A}^{\star}$
- n — a minimal value for what all entries in $\mathtt{A}^{\mathtt{n}}$ are equal to ε.

A must be a square matrix.

Example

```
>> mpm_star(2)
ans =
        0

>> mpm_star(-1)
ans =
     -Inf

>> mpm_star(mpm_zero)
ans =
        0
```

```
>> mpm_star(mpm_one)
ans =
        0

>> mpm_star(mpm_zeros(2, 2))
ans =
        0      Inf
      Inf        0

>> [B, n] = mpm_star([mpm_zero mpm_zero; 6 mpm_zero])
B =
        0      Inf
        6        0
n =
        2

>> [B, n] = mpm_star([mpm_zero −2 −3; 2 10 1; 5 2 mpm_one])
B =
        0       −2       −3
        2        0       −1
        4        2        0
n =
      Inf

>> [B, n] = mpm_star([2 3; mpm_zero −1])
B =
        0      Inf
      Inf     −Inf
n =
      Inf
```

Lets find the maximal solution of $\mathbf{x} = \mathtt{A}\mathbf{x} \vee \mathbf{b}$

```
>> a = 1; b = 2;
>> x = mpm_multi(mpm_star(a), b)
x =
        2

>> mpm_add(mpm_multi(a, x), b) == x
ans =
        1
```

for matrices:

```
>> A = [mpm_zero  mpm_one    mpm_zero;
        mpm_zero  mpm_zero   1;
        mpm_one   mpm_zero   mpm_zero];
>> b = [10;       mpm_zero;  mpm_zero];
>> x = mpm_multi(mpm_star(A), b)
x =
       10
       11
       10
```

Is it correct?

```
>> mpm_add(mpm_multi(A, x), b) == x
ans =
        1
        1
        1
```

See also

mpm_plus, mp_is_pga, mp_is_pgc

---

mpm_zero

(min, +) zero ($-\infty$)
(min, +) neutral element for operation $\vee$

Syntax

```
y = mpm_zero
```

Description

`mpm_zero` returns `Inf`.

Example

```
>> mpm_zero
ans =
      Inf
```

See also

mpm_zeros, mpm_one, mpm_ones, mpm_eye, mpm_add

---

mpm_zeros

(min, +) zeros matrix, vector or scalar

Syntax

```
y = mpm_zeros
y = mpm_zeros(n)
y = mpm_zeros(n, m)
```

Description

- `mpm_zeros` returns `Inf`.
- `mpm_zeros(n)` or `mpm_zeros([n])` returns an `n`–by–`n` matrix of `Inf`-s.
- `mpm_zeros(n, m)` or `mpm_zeros([n m])` returns an `n`–by–`m` matrix of `Inf`-s.

Example

```
>> mpm_zeros
ans =
      Inf

>> mpm_zeros(2)
ans =
      Inf      Inf
      Inf      Inf

>> mpm_zeros(2, 3)
ans =
      Inf      Inf      Inf
      Inf      Inf      Inf
```

See also

mpm_zero, mpm_one, mpm_ones, mpm_eye, mpm_add

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Notation

Here we list some of the acronyms and symbols that occur frequently in this contribution and with which the reader might not be familiar. The numbers in the last column refer to the page on which the symbol or concept is defined.

Acronyms

DES	Discrete Event System	

Sets

$\mathbb{N}$	set of the natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$	
$\mathbb{N}_0$	set of the nonnegative integers	
$\mathbb{R}$	set of the real numbers	
$\mathbb{R}_\varepsilon$	$\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$	
	in the (max, +) $\varepsilon = -\infty$	9
	in the (min, +) $\varepsilon = +\infty$	30
$\mathbb{R}_\varepsilon^n$	set of the column vectors with n components in $\mathbb{R}_\varepsilon$, i.e. $\mathbb{R}_\varepsilon^n \equiv \mathbb{R}_\varepsilon^{n\times 1}$	
$\mathbb{R}_\varepsilon^{m\times n}$	set of the m–by–n matrices with entries in $\mathbb{R}_\varepsilon$	

Matrices and Vectors

$\mathbf{b}$	column vector	
$(\mathbf{b})_i$	i–th element of the column vector $\mathbf{b}$	
$\mathbf{A}^T$	transpose of the matrix $\mathbf{A}$	
$(\mathbf{A})_{ij}$	entry of the matrix $\mathbf{A}$ on the i–th row and the j–th column	
$\mathbf{I}_n$	(max, +) identity matrix, $\mathbf{I}_n \in \mathbb{R}_\varepsilon^{n\times n}$	??
$\boldsymbol{\varepsilon}^{m\times n}$	m–by–n (max, +) zero matrix	??

(max, +) Algebra

$\oplus$	(max, +) algebraic addition	9, ??
$\otimes$	(max, +) algebraic multiplication	9, ??

$\oslash$	(max, +) algebraic division	**??**, 15
e	neutral element for $\otimes : e = 0$	9
ε	neutral element for $\oplus : \varepsilon = -\infty$	9
$\mathbf{A}^\star$	(max, +) star operator (for square matrix), $\mathbf{A}^\star = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \dots$	21
$\mathbb{R}_{\max}$	(max, +) algebra: $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$	9
$\mathbb{R}_\varepsilon^n$	$\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$	9

(min, +) Algebra

$\vee$	(min, +) algebraic addition	30
$\wedge$	(min, +) algebraic multiplication	30
e	neutral element for $\wedge : e = 0$	30
ε	neutral element for $\vee : \varepsilon = +\infty$	30
$\mathbf{A}^+$	shortest path matrix (min, +) plus operator (for square matrix), $\mathbf{A}^+ = \mathbf{A}^1 \vee \mathbf{A}^2 \vee \dots$	30
$\mathbf{A}^\star$	(min, +) star operator (for square matrix), $\mathbf{A}^\star = \mathbf{I} \vee \mathbf{A}^+ = \mathbf{I} \vee \mathbf{A}^1 \vee \mathbf{A}^2 \vee \dots$	
$\mathbb{R}_{\min}$	(min, +) algebra: $\mathbb{R}_{\min} = (\mathbb{R}_\varepsilon, \vee, \wedge)$	30
$\mathbb{R}_\varepsilon^n$	$\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$	30

Miscellaneous

$\mathcal{G}(\mathbf{A})$	precedence graph of the matrix $\mathbf{A}$	17
$tr(\mathbf{A})$	trace of the matrix $\mathbf{A}$	15
λ	maximal cycle mean of $\mathcal{G}(\mathbf{A})$ (largest) eigenvalue of $\mathbf{A}$	18 21

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